Twistor Theory Project

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Abstract: This project aims to introduce twistor theory at the level of a masters student in theoretical physics. We focus on the geometrical aspects of twistor theory, presenting the incidence relations and how they relate the geometry of complex Minkowski space, $\mathbb{M}_{\mathbb{C}}$, to projective twistor space, $\mathbb{PT}$: a point in $\mathbb{M}_{\mathbb{C}}$ corresponds to a line in $\mathbb{PT}$; and two intersecting lines in $\mathbb{PT}$ (which define a point in $\mathbb{PT}$) correspond to null separated points in $\mathbb{M}_{\mathbb{C}}$. The main result is a reasonably detailed presentation of the linear Penrose transform; an isomorphism which relates the solutions of zero-rest-mass field equations with helicity n and the first Čech cohomology group $\check{H}^1(\mathbb{PT}^{\pm}; \mathcal{O}(-n-2))$ on projective twistor space. We also present several other neat twistor theory results, including a demonstration of how to encode the conformal structure of the spacetime in twistor space.

Acknowledgements

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Secondly, like to thank Peter Ferguson & Ryan Cullinan for undertaking the task of proof reading the draft and giving their advice on how best to communicate the results and pointing out the many grammatical errors present.

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Lastly I would like to thank Durham University for providing me the opportunity to work on this report and their continued support whenever needed. Specifically I would like to thank David Cerdeño and Nabil Iqbal for running the MSc course and putting up with us students!

All the unreferenced figures that appear in this document were drawn by myself for this project (having sometimes been inspired by figures seen elsewhere).

Richie Dadhley

Dedication

I would like to dedicate this work to my late grandparents,

Satwant Dadhley & Stanley Ives.

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0 Introduction

Twistor theory is a relatively young theory, with its first two papers being published in the late 1960's, [1, 2]. Penrose's initial hope for twistor theory was to provide a novel approach to the problems of quantum gravity. However, as it happened, by the 1980's the topic sadly hit a road block for its applications to physics. This did not stop people studying twistor theory itself, and over the following 20 years mathematicians really took the reins, using the techniques to help better understand geometrical topics. The physics community once again found inspiration in twistor theory, when none other then Witten published a paper [3] in 2004 demonstrating the links between the calculation of *entire* tree-level scattering amplitudes of Yang-Mills theory in 4-dimensional spacetime and twistor theory. Since then twistor theory has found applications in both a purely mathematical setting as well as in the more physics oriented goal of understanding quantum field theory (QFT).

The 'sales pitch' of the twistor programme is to try replace the familiar ideas of a background spacetime with a new space, namely twistor space, with the idea that the 'fundamental' properties of spacetime can then be viewed as a repercussion of structures on twistor space. The connection between these two spaces is given by the *incidence relations*. We can loosely compare this to the analogy of introducing momentum space as the Fourier transform of spacetime [4].

As the above has highlighted, twistor theory naturally treads the line between precise, yet rather abstract, mathematics and the, perhaps more intuitive, realms of theoretical physics. This naturally means that few graduate students study twistor theory, as they tend to lean too heavily to one side of this delicate line. This then means that an introductory text in twistor theory must 'pull in' both sides, first presenting the mathematics in order to then demonstrate its connections to the physics. Naturally this would make such a text anything but 'nutshell' in size![1] This project aims to achieve such a balancing act, however it must be said at this point that the 'leaning' is naturally reflective of the author's own interests, while keeping a page restriction in mind. We therefore present a lot of the material in a semi-rigorous manner, often delving into the topics of algebraic and differential geometry, however we must unfortunetly gloss over some of these details. Perhaps the biggest example of this is a proper discussion of so-called principal G-bundles, which play a vital role in the construction of spinor fields. A brilliant discussion of this material can be found via the online course [5], and the interested reader is directed there.

The ultimate aim of this project is to present twistor space and the previously mentioned incidence relation. We then want to study the novel geometry of this construction and how it links to the geometry of spacetime itself. We will also discuss how ideas such as conformal field theory (CFT) is captured in twistor space. The main physics goal is to present the infamous *Penrose transform*, which relates the solutions to zero-rest-mass field equations (e.g. Maxwell's equations) to a mathematical construction on twistor space.

The project is laid out as follows. First we recap/discuss the notions of tensors and the action of the Lorentz group. At the end of this chapter we will briefly introduce bundles in order to be able to use the terminology going forward. Then in Chapter 2 we present the double covering map, which allows us to introduce the connection between vectors and spinors. This will give us our first 'taste' of twistor theory, however it is only in Chapter 3 that we will introduce twistor space itself, along with the zero-rest-mass equations and the incidence relation. Here we will study the geometrical links between the two spaces, as well as how CFT is encoded in twistor terms. Chapter 4 is then an almost purely mathematical chapter, dedicated to introducing *sheaf cohomology*, which will prove vital for demonstrating the Penrose transform in Chapter 5. We then finally conclude the project, suggesting further reading for the interested reader.

[1] Unless the length of the popular 'blah in a nutshell' books are taken as a reference!

1 | Tensors & The Lorentz Group

We start the main content of this project by demonstrating an elementary correspondance which forms the basis for the construction of twistor theory: the *double covering of the Lorentz group by the spin group.*[1] We should point out that this correspondance of course has applications outside of twistor theory itself, perhaps most notably in supersymmetry. This will also be true of a reasonable amount of the other material presented in this project, and we shall try as best we can to be clear on the applicability of the results as we go.

In order to demonstrate our double covering map, we obviously need to understand the target space, the Lorentz group, properly and so this chapter is dedicated to such a discussion. The reader should be fairly familiar with the construction of tensors and the action of the Lorentz group that follows, but we provide it here for clarity and also to clear up some conventions.

More details on the material presented here can be found via [5–8], as well as a plethora of other books/courses on differential geometry.

1.1 Recap On Tensors

1.1.1 Vectors

The most basic (or at least most familiar and geometrically intuitive) example of a tensor is a vector. Geometrically we think of a vectors as little 'arrows' that we can add, subtract and scale. Indeed this is the basic idea behind the definition of a vector space. Vector spaces are their own abstract mathematical constructions, but they become relevant when discussing spacetimes as the *tangent space* to a point $p \in \mathcal{M}$, where $\mathcal{M}$ is our *manifold*, are vector spaces. We denote the tangent space to $p \in \mathcal{M}$ simply as $T_p\mathcal{M}$, and will typically denote elements as, e.g., $X_p \in T_p\mathcal{M}$. We denote the set of all tangent vectors to $\mathcal{M}$ as $T\mathcal{M}$, with no subscript.

Now it is important to note that all of these tangent spaces are completely independent and know nothing about each other. That is, it is utter nonsense to write something like $X_p + Y_q$, where $X_p \in T_p\mathcal{M}$ and $Y_q \in T_q\mathcal{M}$. In other words each vector is a *completely* local quantity, see Figure 1.1. However we know from our experiences in physics that we would also like a global (or at least in some open region) notion of vectors. That is, the electric field[2] is not defined at a single point but over some region. This motivates us to introduce vector *fields*, which are essentially just a collection of different vectors, each at a different point. We will denote vector fields without the subscripts, i.e. X is a vector field while Y_p is merely a vector. It is hopefully clear that a vector field is a subset of $T\mathcal{M}$. We will later denote the set of vector fields as $\Gamma T\mathcal{M}$, but will avoid using this notation until then.

The obvious question to ask is "how do we pick our vectors in order to get a nice vector field?" The most satisfying answer to this question is given in terms of the *bundle* formalism, and we shall touch on this briefly at the end of this chapter. For now we just lean on intuition of "we want them to follow on from each other nicely".

Before moving on to discuss other types of tensor fields, we first want to recall[3] an important claim.

[1] We will also use the language of *homomorphisms* to describe such maps. That is we want to find 2-to-1 homomorphism Ψ : $\text{Spin}(n) \to SO(n)$.

[2] Technically the electric field is an example of something called a *1-form*, but this distinction is not important right now.

[3] It is assumed the reader is familiar with this from the context of general relativity. For this reason the proof (which simply follows from considering a chart containing $p \in \mathcal{M}$) is omitted.

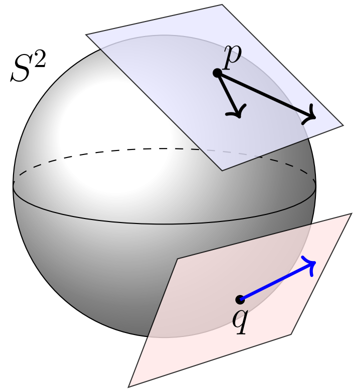


Figure 1.1: *Tangent planes at two points $p, q \in \mathcal{M} = S^2$. The two black arrows indicate two different tangent vectors in $T_p\mathcal{M}$, and so can be added, however the blue arrow is in $T_q\mathcal{M}$ and so knows nothing about the black arrows. Figure taken from my notes on [7].*

Claim 1.1.1. Any tangent space to any manifold has the same dimension as the manifold itself, i.e.

$$\dim T_p\mathcal{M} = \dim \mathcal{M} \qquad \forall p \in \mathcal{M}. \tag{1.1}$$

1.1.2 Cotangent Vectors

The next important example of a tensor is what is known as a *covector*. These are also vectors in the vector space sense, i.e. we can add and scale them, however they are given by the *dual* of some other vectors. In the case of tangent vectors they define *cotangent vectors* as follows.

> **Definition.** [Cotangent Vector] Let $X_p \in T_p\mathcal{M}$, where $\mathcal{M}$ is a real manifold.[4] Then a covector at $p \in \mathcal{M}$, ω_p, is a linear map
> $$\omega_p : X_p \to \mathbb{R}.$$
> We denote the space of cotangent vectors to $p \in \mathcal{M}$ as $T_p^*\mathcal{M}$. The full set of all cotangent vectors are then denoted $T^*\mathcal{M}$.

We can form covector fields by taking a collection covectors at different points. Each covector maps the corresponding tangent vector to a real number, and so the final result is a collection of real numbers. This is a scalar field over $\mathcal{M}$, the set of which is denoted $C^\infty(\mathcal{M})$. That is

$$\omega : X \to f \in C^\infty(\mathcal{M}).$$

Claim 1.1.2. If our manifold is finite dimensional (and so $T_p\mathcal{M}$ is finite dimensional, by Equation (1.1)) then

$$T_p^{**}\mathcal{M} = T_p\mathcal{M}. \tag{1.2}$$

Again it is assumed the reader is familiar with this claim and so the proof is omitted. It is included as it allows us to see that we can equally view tangent vectors as maps acting on cotangent vectors,

$$X_p : \omega_p \to \mathbb{R}.$$

1.1.3 Higher Valence Tensors

Now that we have both vectors and covectors we can easily form higher valence tensors via the tensor product.

> **Definition.** [Tensors] Let $(V, +, \cdot)$ be a vector space, then we define a (r, s)-tensor by
> $$T = \underbrace{V \otimes ... \otimes V}_{r\text{-times}} \otimes \underbrace{V^* \otimes ... \otimes V^*}_{s\text{-times}}, \tag{1.3}$$

which can be viewed as a map

$$T : \underbrace{V^* \times ... \times V^*}_{r\text{-times}} \times \underbrace{V \times ... \times V}_{r\text{-times}} \to \mathbb{R}.$$

We can modify the map above by only filling in some of the tensors entries. That is consider a $(1,1)$ tensor

$$T : V^* \times V \to \mathbb{R}.$$

If we only fill the first slot (the V^* one), we are left with something that can still eat an element of V and produce a real number. This is, of course, just a covector. This idea clearly extends to other types of tensors, which we summarise by saying "only filling certain indices will give us some other, lower rank[5] tensor". We shall return to this idea in a moment when talking about the metric.

We now point out that we have been careful above to try demonstrate the generality of the notion of a tensor. That is Equation (1.3) holds for *any* vector space, and need not be the tangent vector space described above. This will be important later as it will mean that we can construct higher valence spinor tensors without having to introduce any new technology.

If we consider the case of our tangent and cotangent vectors, we can form tensor fields over our manifold again with the intuitive notion of taking a tensor at every point.

1.1.4 Abstract Index Notation

The constructions above were all done coordinate free, that is we haven't used charts in order to construct our tensor fields. However we know from our experiences with general relativity that in order to do basically any useful calculation we really need the components of our tensors. However we also know that we should be weary of using components as these make reference to a *specific* coordinate chart, and this may lead us to incorrectly read off chart-dependent results as physical. The most famous example being the coordinate singularity of an event horizon in Schwarzschild coordinates.

For these reasons, we shall often work with Penrose's so-called *abstract index notation*[9]. The basic idea is that we write down indices on our tensors but do not use any *specific* coordinate system. This is different to writing down the components of a tensor, as components are defined w.r.t. a given coordinate system. We will use lowercase Latin indices for this abstract index notation, and if at any point we make reference to a specific coordinate system we will switch to the usual Greek indices.

We also adopt the usual conventions of GR where vectors come with an upper index, X^a, and covectors with a lower index ω_a. An (r,s) tensor then has r upper indices and s lower indices,[6] $T^{a_1...a_r}{}_{b_1...b_s}$. We also adopt Einstein's summation convention, two repeated indices, one upper and one lower are summed over.

There are three important algebraic operators we can conduct on higher valence tensors. Given an (r,s) tensor:

(a) Contraction: if both $r, s \neq 0$ we can *contract* one upper index with a lower one to produce a $(r-1, s-1)$ tensor,

$$S^{a_2...a_r}{}_{b_2...b_s} := T^{ca_2...a_r}{}_{cb_2...b_s}.$$

(b) Symmetrisation: if $r \geq 2$ we can symmetrise $n \leq r$ of these indices via (omitting the lower indices)

$$T^{(a_1...a_n)a_{n-1}...a_r} := \frac{1}{n!}\sum_{\sigma\in\mathfrak{G}} T^{a_{\sigma(1)}...a_{\sigma(n)}a_{n-1}...a_r},$$

where $\mathfrak{G}$ is the set of all permutations of n elements. We can similarly symmetrise the lower indices, provides $s \geq 2$.

[5]The *rank* of an (r,s) tensor is the sum $s+r$.

[6]We shall also always assume that we take our tensor products so that our upper indices appear first, and then the lower ones. This is basically done by isomorphisms, e.g. $V^* \otimes V \otimes V^* \cong V \otimes V^* \otimes V^*$.

(c) Antisymmetrisation: Similarly to symmetrisation, we can antisymmetrise by including the *sign* of the permutation[7]

$$T^{[a_1\dots a_n]a_{n-1}\dots a_r} := \frac{1}{n!}\sum_{\sigma\in\mathfrak{G}}(-1)^{\sigma}T^{a_{\sigma(1)}\dots a_{\sigma(n)}a_{n-1}\dots a_r},$$

An important example of a higher valence tensor for us will be an antisymmetrised $(2,0)$ tensor,

$$T^{[ab]} = \frac{1}{2}\big(T^{ab} - T^{ba}\big) \tag{1.4}$$

which we shall henceforth call a *bivector*. One important property of bivectors for us is the notion of a *simple* bivector.

Definition. A bivector T^{ab} is said to be *simple* iff we can write it as an antisymmetric product of two vectors, i.e.

$$T^{ab} = U^{[a}V^{b]}.$$

1.1.5 The Metric

We *now* want to introduce the metric. We emphasise "now" as we wish to point out that the construction of tensor fields above made no use of the metric at all. We will emphasise this a few times, as the need for a metric is one of the key distinguishing properties between a tensor field and a spinor field.

Definition. [Metric] The metric, g, is a $(0,2)$ tensor field on a manifold $\mathcal{M}$ satisfying

(a) Symmetry: $g(X,Y) = g(Y,X)$, and

(b) Non-degeneracy: there exists a $(2,0)$ tensor field which is the inverse of g.

(c) Bilinear

In abstract index notation we denote the metric by g_{ab}, the inverse by g^{ab}. The two conditions above are then written

(a) $g_{ab} = g_{ba}$, and

(b) $g^{ac}g_{cb} = \delta^a_b$.

The metric is what allows us to measure lengths and angles on our manifold. In this way it is what gives 'shape' to our spaces; it differentiates two manifolds with the same topology, e.g. a potato vs. a round sphere.

As the metric is a $(0,2)$ tensor field, it eats two vectors. However we said above that if we only fill one of the slots we again obtain a tensor, in particular a covector. In this way we can "lower the index" on a vector X^a to give us a covector $X_b := g_{ba}X^a$. We can use this construction *define* the covector space via $(X^a)^* := X_a$. We can equally lower the index on higher valence tensors, provided $r \neq 0$ (otherwise there are no upper indices to lower!). Similarly the inverse metric g^{ab} can be used to "raise" the index on a (r,s) tensor with $s \neq 0$.

When considering spacetime, we take our metric to have *Lorentzian signature*, that is in an *orthonormal frame* we have

$$g_{ab} = \mathrm{diag}(1,-1,-1,-1),$$

where we have assumed we are considering a 4-dimensional spacetime, and fixed our convention to be "mostly minus".

The key point we need here is that the metric allows us to define a Lorentzian inner product on our tangent spaces. This allows us to categorise vector fields as being either

(a) Timelike: $g(X,X) > 0$,

[7]See, e.g., appendix A of [6] if this is unfamiliar.

(b) Spacelike: $g(X,X) < 0$, or

(c) Lightlike/Null: $g(X,X) = 0$.

These give us the familiar notion of a light-cone/null-cone, see Figure 1.2. It is important to note that the null-cone, being defined by its action on a vector, lives in the tangent space. That is, in general, we have a different null cone for each $T_p\mathcal{M}$.

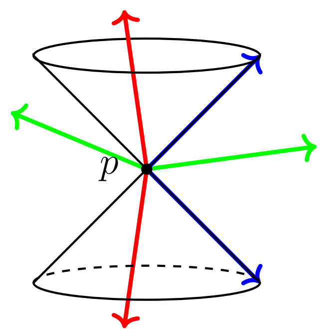


Figure 1.2: *Example of a null-cone structure at a point $p \in \mathcal{M}$. The red arrows are timelike vectors, green arrows spacelike ones and the blue ones null. Note that, as it stands, we have no way of saying which cone is 'the future' and which is 'the past'. Figure taken (and edited) from my notes on [7].*

1.2 Forms

Most of the above material should have been more of a recap than introduction of new material. We now introduce a structure that is not assumed to be previously understood; the notion of *differential forms*.

The easiest/quickest definition we can give of a differential form is a *totally antisymmetric* $(0, s)$ tensor field. As we have $r = 0$, we normally categorise forms by the s value, which we relabel p. That is a *p-form* is a totally antisymmetric $(0,p)$ tensor field. We denote the set of all p-forms on $\mathcal{M}$ by $\Lambda^p\mathcal{M}$.[8]

Differential forms are vastly interesting and there is a lot we can do with them, but here we shall just summarise the important tools for us. The first thing we note is that we have already encountered two types of differential forms, namely:

1. 0-forms: these are simply functions, $\Lambda^0\mathcal{M} = C^\infty(\mathcal{M})$, and
2. 1-forms: these are our covectors, $\Lambda^1\mathcal{M} = T^*\mathcal{M}$.

1.2.1 Top Forms & Volume Forms

The next important thing we note is that, if our manifold has dimension d, we can't have a non-vanishing $(p > d)$-form, as we would necessarily need to have at least two of the indices the same, and then antisymmetry would set it to zero. In other words, we have to cap $0 \leq p \leq d$. We therefore see that d-forms are rather special, and we give them the name *top* forms. A subcategory of top forms are *volume* forms, defined as follows.

Definition. [Volume Form] Let $\mathcal{M}$ be a d dimensional manifold. Then a top form Ω is called a *volume form* if it is nowhere vanishing, $\Omega|_p \neq 0$ for all $p \in \mathcal{M}$.

Volume forms allow us to define a preferred set of *frames* — for now we take a frame to just to be a basis in each tangent vector space — on a given manifold, which is highlighted in the next definition.

[8] The p here does *not* mean a point $p \in \mathcal{M}$, but rather the rank p. That is $\Lambda^p\mathcal{M} = \bigwedge^p (T^*\mathcal{M})$, where $\bigwedge^p$ denotes our total antisymmetrisation over the p copies of $T^*\mathcal{M}$.

Definition. [Right-Handed Frame] A basis $(X_1, ..., X_d)$ of tangent vectors is called right-handed if

$$\Omega(X_1, ..., X_d) > 0. \tag{1.5}$$

We can similarly define a left-handed frame, but with $\Omega < 0$. This notion of a right- vs. left-handed frames are exactly those we are familiar with from vector calculus.

Besides giving us right- and left-handed frames, volume forms also allow us to introduce an *orientation* of a manifold, which we now define.

Definition. [Orientation Of A Manifold] A manifold $\mathcal{M}$ is said to be *orientable* if we can restrict the chart transition maps in such a way that the determinant Jacobian only takes one sign, i.e.

$$\det\left(\frac{\partial y^\nu}{\partial x^\mu}\right) > 0 \qquad \textbf{OR} \qquad \det\left(\frac{\partial y^\nu}{\partial x^\mu}\right) < 0$$

for any two overlapping charts (U, x) and (V, y).

We then have the following proposition.

Proposition 1.2.1. *A manifold is orientable iff it admits a volume form.*

Now note that, for a given manifold, *all* top forms are related to each other by the multiplication by a scalar field. That is if $\omega, \xi \in \Lambda^d\mathcal{M}$ then there *must* exist a $f \in C^\infty(\mathcal{M})$ such that

$$\omega = f\xi.$$

This is important as it tells us that we can relate our top forms to the d-dimensional Levi-Civita symbol

$$\epsilon_{a_1...a_d} = \begin{cases} 1 & \text{if } (a_1, a_2, ..., a_d) \text{ is an } \textit{even} \text{ permutation of } (1, 2, ..., d), \\ -1 & \text{if } (a_1, a_2, ..., a_d) \text{ is an } \textit{odd} \text{ permutation of } (1, 2, ..., d), \\ 0 & \text{otherwise.} \end{cases}$$

As we are working with a Lorentzian metric, when we raise all the indices on the Levi-Civita symbol we get a sign change, that is

$$\epsilon^{a_1...a_d} = \begin{cases} -1 & \text{if } (a_1, a_2, ..., a_d) \text{ is an } \textit{even} \text{ permutation of } (1, 2, ..., d), \\ 1 & \text{if } (a_1, a_2, ..., a_d) \text{ is an } \textit{odd} \text{ permutation of } (1, 2, ..., d), \\ 0 & \text{otherwise.} \end{cases}$$

Now the Levi-Civita symbol is not a genuine tensor but a tensor *density*. This basically means that it is only a tensor once we weight it with the square root of the determinant of the metric. We can therefore relate our volume form to the Levi-Civita symbol by

$$\Omega_{a_1...a_d} := \sqrt{g}\epsilon_{a_1...a_d},$$

where $g := |\det g_{ab}|$. The antisymmetry is obviously inherited from $\epsilon_{a_1...a_d}$, and is non-vanishing because the metric is globally non-degenerate (i.e. $\det g_{ab} \neq 0$). The completely raised version of this volume form is then given by

$$\begin{aligned} \Omega^{b_1...b_d} &= \sqrt{g} g^{b_1 a_1} ... g^{b_d a_d} \epsilon_{a_1...a_d} \\ &= \sqrt{g} g^{-1} \epsilon^{b_1...b_d} \\ &= \sqrt{g^{-1}} \epsilon^{b_1...b_d} \end{aligned}$$

where we have used

$$g^{b_1 a_1} ... g^{b_d a_d} \epsilon_{a_1...a_d} = \det g^{ab} =: g^{-1}$$

and the determinant rules

$$\det A \det B = \det(AB) \quad \text{and} \quad \det A^{-1} = \frac{1}{\det A}.$$

Finally, using the identity

$$\epsilon_{a_1...a_d}\epsilon^{a_1...a_d} = -d!,$$

we obtain

$$\Omega_{a_1...a_d}\Omega^{a_1...a_d} = \epsilon_{a_1...a_d}\epsilon^{a_1...a_d} = -d!$$

In this way we can identity the choice of orientation with the specification of the d-dimensional Levi-Civita symbol.

Next note that using that $\sqrt{g} > 0$, we see that Equation (1.5) is equivalent to

$$\epsilon_{a_1...a_d}X^1...X^d > 0,$$

and so we can also link our notion of right- and left-handed to the Levi-Civita symbol. If we take an *orthonormal* right-handed frame we get

$$\epsilon_{a_1...a_d}X^1...X^d = \sqrt{g^{-1}}.$$

For our 4-dimensional Minkowski (so $\sqrt{g} = 1$) spacetime we denote our orthonormal right/let-handed frame as

$$\epsilon_{abcd}T^aX^bY^cZ^d = \pm 1. \tag{1.6}$$

Time Orientation

As the names have suggested, orientation and handedness essentially allow us to completely orientate our frames in spacetime, and so we can differentiate 'future' from 'past'. This is done by choosing one of the light cones in Figure 1.2 to be a "future null-cone" and the other is then the "past null-cone". Mathematically this is done by introducing a so-called *time-orientation* on an oriented Lorentzian manifold. This is a smooth vector field T that

(a) is no-where vanishing, and

(b) time-like, $g(T,T) > 0$.

Smooth essentially means that our vector field doesn't suddenly "jump" as we move from one point $p \in \mathcal{M}$ to the next $q \in \mathcal{M}$. We demonstrate this pictorially in Figure 1.3.

1.2.2 Hodge Dual

We now need to introduce an interesting and useful operation on forms. It goes by the name of *Hodge dual*, denoted $\star$.

> **Definition.** [Hodge Dual] Let $\mathcal{M}$ be a d-dimensional manifold and $\omega \in \Lambda^p\mathcal{M}$ be a p-form, for any $0 \leq p \leq d$. Then the Hodge dual is defined as a linear mapping
>
> $$\star : \Lambda^p\mathcal{M} \to \Lambda^{d-p}\mathcal{M},$$
>
> which in components reads
>
> $$\begin{aligned}(\star\omega)_{a_1...a_{d-p}} &:= \frac{1}{p!}\sqrt{g}\epsilon_{a_1...a_{n-p}b_1...b_p}\omega^{b_1...b_p} \\ &= \frac{1}{p!}\sqrt{g}\epsilon_{a_1...a_{n-p}}{}^{b_1...b_p}\omega_{b_1...b_p},\end{aligned} \tag{1.7}$$
>
> where the second line simply follows from the fact that we can freely exchange the our raised and lowered indices.

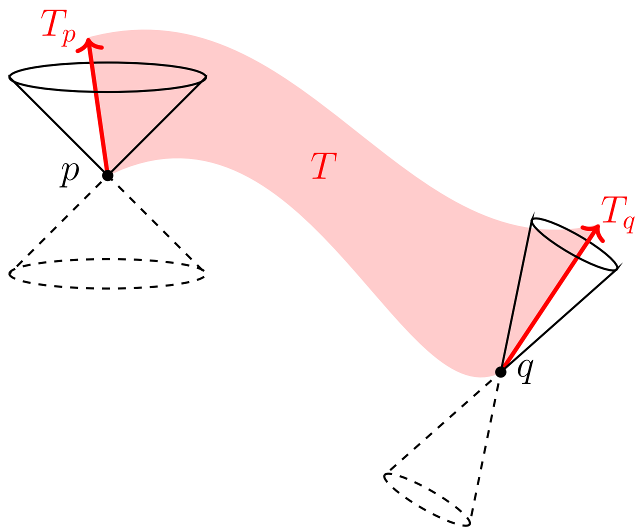


Figure 1.3: *Pictorial representation of the relativistic spacetime. The metric g produces a double cone structure in the tangent plane to each point of the manifold. In order to differentiate the two cones, a smooth vector field T is introduced in such a way that, at each point $p \in M$, the vector $T_p \in T$ points within one of the two cones associated to that point. This cone is then identified as the 'future' relative to that point. The smoothness of T (indicated by the shaded region) ensures a nice transition from the 'future' of one cone to another. Solid lined cones indicate the chosen 'future' cones and dashed the 'past' cones. Figure taken from my notes on [7].*

From here we can readily check that

$$\star^2 \omega := \star(\star\omega) = (-1)^{p(d-p)+1}\omega. \tag{1.8}$$

Why is this useful for us? Well we note that we are dealing with a 4-dimensional manifold and so 2-forms — which are the lowered versions of our bivectors introduced above, Equation (1.4) — are mapped to 2-forms under the Hodge dual. On top of this they obey

$$\star^2 F = -F, \qquad F \in \Lambda^2\mathcal{M},$$

and so we conclude that 2-forms are eigenvectors of the Hodge dual with eigenvalues $\pm i$. The eigenvectors with eigenvalue $+i$ are called *self-dual* and those with $-i$, *anti-self-dual*. In other words:

The Hodge star operator gives us an involution on 2-forms, which induces a decomposition

$$\Lambda^2\mathcal{M} = \Lambda^2_+\mathcal{M} \oplus \Lambda^2_-\mathcal{M}, \tag{1.9}$$

where $\Lambda^2_+\mathcal{M}$ is the space of self-dual 2-forms and $\Lambda^2_-\mathcal{M}$ the space of anti-self-dual 2-forms. These two spaces only depend on the conformal class[a] of g [11].

[a] It is assumed the reader is familiar with the notion conformal symmetries. More information can be found in, e.g., Simmons-Duffin's TASI notes, [10].

1.2.3 Exterior Derivative

Next we want to introduce what is known as the *exterior derivative*, denoted d. In its most basic sense, it is a map that increases the rank of a form by one:

$$d : \Lambda^p \mathcal{M} \to \Lambda^{p+1}\mathcal{M}. \tag{1.10}$$

To give a more complete definition of d, we need to introduce the wedge product. This is simply an antisymmetric tensor product on forms,

$$\alpha \wedge \beta := \alpha \otimes \beta - \beta \otimes \alpha,$$

which, if $\alpha \in \Lambda^p\mathcal{M}$ and $\beta \in \Lambda^q\mathcal{M}$, is clearly a map to $\Lambda^{p+q}\mathcal{M}$. It follows immediately from our discussion of top forms that we require $p + q \leq \dim \mathcal{M}$, if we want a non-vanishing result. We then have the following definition

Definition. [Exterior Derivative] The *exterior derivative* is the mapping Equation (1.10) satisfying:

(i) Linearity: Let $\alpha, \beta \in \Lambda^p\mathcal{M}$ be any p-forms, then $d(\alpha + \beta) = d\alpha + d\beta$.

(ii) Graded Leibniz: Let $\alpha \in \Lambda^p\mathcal{M}$ and $\beta \in \Lambda^q\mathcal{M}$, then $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^p \alpha \wedge d\beta$.

(iii) Let $f \in \Lambda^0\mathcal{M} \cong C^\infty(\mathcal{M})$ and $X \in \Gamma T\mathcal{M}$ be a smooth function and vector field, respectively, then[9]

$$df(X) := X(f).$$

(iv) Nilpotency ("It squares to zero"): For any $\alpha \in \Lambda^p\mathcal{M}$, with $0 \leq p \leq \dim \mathcal{M}$

$$d^2\alpha := d \circ d\alpha = 0. \tag{1.11}$$

Claim 1.2.2. The exterior derivative map is unique.

Proof. Omitted. See, e.g., the proof of Theorem 2.5 in [12]. □

The first three conditions are obviously important and useful, however the most important for us will be condition (iv), and will be the grounding for the next subsection.

1.2.4 deRham Cohomology

We will develop a more rigorous definition of cohomology in Chapter 4, but it is instructive for us at this point to give a quick introduction via *deRham cohomology*.

We start by introducing *closed* p-forms, which we denote $Z^p(\mathcal{M}) \subseteq \Lambda^p\mathcal{M}$. These are basically the forms that lie in the kernel of d, i.e. $\alpha \in Z^p(\mathcal{M})$ iff $\alpha \in \Lambda^p\mathcal{M}$ and $d\alpha = 0$. Now it follows immediately from (iv) that if there exists a $\beta \in \Lambda^{p-1}\mathcal{M}$ such that $\alpha = d\beta$, then α is closed. We call such p-forms *exact* and denote them $B^p(\mathcal{M})$. It is hopefully clear that these sets obey

$$B^p(\mathcal{M}) \subseteq Z^p(\mathcal{M}) \subseteq \Lambda^p\mathcal{M}.$$

The aim of deRham cohomology is to ask the question "how close is the first equality?" That is, it is a measure of the extent to which a closed form fails to be exact. Mathematically, the p-th deRham cohomology is given by the quotient space

$$H^p_{dR}(\mathcal{M}) := Z^p(\mathcal{M})/B^p(\mathcal{M}), \tag{1.12}$$

i.e. two p-forms that differ only by an exact form as equivalent:[10]

$$\alpha \sim \beta \qquad \iff \qquad \alpha = \beta + d\gamma,$$

[10] See, e.g., Appendix A.1 of [6] for a discussion of equivalence relations.

where $\alpha, \beta \in \Lambda^p\mathcal{M}$ and $\gamma \in \Lambda^{p-1}\mathcal{M}$.

So why is the deRham cohomology interesting? The answer is that it tells us interesting things about the topology of the manifold $\mathcal{M}$, in particular it tells us about the number of connected components and the number of 'holes' in $\mathcal{M}$. There are several ways to show these points,[11] but here we offer a nice geometrical explanation, based on the one given in [13].

We can think of the exterior derivative as giving the boundary of the thing it acts on. So if $f \in C^\infty(\mathcal{M})$ is some smooth function on $\mathcal{M}$, then df are the contour lines of f. We can easily convince ourselves that if $\dim \mathcal{M} = n$, these contours are $(n-1)$-dimensional submanifolds of $\mathcal{M}$. Extending this argument, a closed p-form is a boundariless $(n-p)$-dimensional submanifold in $\mathcal{M}$. Another helpful example are volume forms: these are naturally closed (as we cannot have a $(\dim \mathcal{M} + 1)$-form), and are geometrically a 0-dimensional submanifold, i.e. an allocation of a number, the *measure*, to each point in $\mathcal{M}$.

From here it is intuitively clear that the 0-th deRham cohomology tells us the number of connected components in $\mathcal{M}$. That is, the number of connected components of an n-dimensional manifold is exactly the number of boundariless n-dimensional submanifolds.

We can intuitively ask about higher dimensional cohomology classes by then rewording condition (iv) as "the boundary of a boundary vanishes." The question of "how close is the first equality?" then becomes "are there any boundariless $(n-p)$-dimensional submanifolds of $\mathcal{M}$ that are *not* the boundary of a $(n-p+1)$-dimensional submanifold?" This may sound like confusing wording at first, however we then note an important idea: if something is the boundary of another space, it naturally can be contracted away, namely by shrinking it through the latter space.[12] So our deRham cohomology question finally becomes "how many, topologically different, *non*-contractible $(n-p)$-dimensional submanifolds are there on $\mathcal{M}$?" This is basically the question of "how many 'holes' does $\mathcal{M}$ contain?"

We can therefore 'prove'[13] results like

$$H^p_{dR}(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{if } p = 0, \text{ and} \\ 0 & \text{otherwise} \end{cases}$$

$$H^p_{dR}(S^n) = \begin{cases} \mathbb{R} & \text{if } p = 0, n, \text{ and} \\ 0 & \text{otherwise} \end{cases}$$

$$H^p_{dR}(T^n) = \begin{cases} \mathbb{R} & \text{if } p = 0, n \\ \mathbb{R}^n & \text{if } p = 1, n-1, \\ 0 & \text{otherwise} \end{cases}$$

where T^n is the n-torus.

1.3 Lorentz Transformations

We shall now recap Lorentz transformations, highlighting some important points.

The Lorentz group for our 4-dimensional Lorentzian manifold is $L = O(1,3)$, and is defined as the group of endomorphisms[14] on our tangent planes that preserve the metric, i.e. $\Lambda^a{}_b \in L$ iff

$$\Lambda^a{}_b \Lambda^c{}_d \eta_{ac} = \eta_{bd}. \tag{1.13}$$

Considering the Lie algebra, $\mathfrak{o}(1,3)$, we can readily show that $\dim L = 6$, given by the antisymmetric 4×4 matrices $\omega_{ab} = -\omega_{ba}$.

Vectors transform in the fundamental representation of the Lorentz group. This just means that Lorentz transformations act on vectors simply as

$$\Lambda^a{}_b : V^b \mapsto \Lambda^a{}_b V^b.$$

[11] The most mathematically satisfying probably being through the use of homotopy, see, e.g. [6].

[12] We have to keep track of the orientation of the submanifold. This allows us to 'cancel' the two circles that form the boundary of a annulus, for example.

[13] Of course much more rigorous proofs exist, most commonly using exact sequences and the Mayer-Vietoris sequence. We will comment on this a bit more in Chapter 4.

[14] That is maps from the space to itself.

Putting this together with Equation (1.13), we see that Lorentz transformations are exactly those spacetime length preserving transformations on vectors. That is if

$$\widetilde{V}^a = \Lambda^a{}_b V^b \qquad \text{then} \qquad g(\widetilde{V}, \widetilde{V}) = g(V, V).$$

As we only have $O(1,3)$, not $SO(1,3)$, we have both $\det \Lambda = \pm 1$. From the fact that

$$\Lambda^a{}_e \Lambda^b{}_f \Lambda^c{}_g \Lambda^d{}_h \epsilon_{abcd} = \det \Lambda \epsilon_{efgh},$$

we see that we can decompose our Lorentz group into orientation preserving and orientation reversing transformations. That is we have $L = L_+ \cup L_-$ where $\Lambda \in L_+$ implies $\det \Lambda = 1$, and similarly for L_-. We can further decompose these Lorentz transformations by how to effect the right- vs. left-handedness: the $\Lambda^0{}_0$ component tells us how the time-axis is effected; if $\Lambda^0{}_0 > 0$ it is left unchanged, while if $\Lambda^0{}_0 < 0$ it flips. From Equation (1.6) we see this corresponds to what happens to the right/left-handedness. So in total we have a decomposition of the Lorentz group as

$$L = L_+^\uparrow \cup L_+^\downarrow \cup L_-^\uparrow \cup L_-^\downarrow, \tag{1.14}$$

where $\uparrow$ means $\Lambda^0{}_0 > 0$ and $\downarrow$ means $\Lambda^0{}_0 < 0$. Only one of these decompositions forms a proper subgroup (as it contains the identity) and that is, of course, $L_+^\uparrow$. We refer to this as the *proper orthochronus Lorentz group*. This subgroup will be very important to use in the next chapter.

1.4 Bundles

We conclude this chapter with a brief introduction to bundles, as they are the natural setting needed to discuss spinor fields and a lot of the language translates into twistor theory. We first provide a list of definitions and then tie them all together in Figure 1.4.

Definition. [Bundle] A *smooth bundle* is a triple $(E, \pi, \mathcal{M})$, where E and $\mathcal{M}$ are smooth manifolds, known as the total space and base space, respectively, and $\pi : E \to \mathcal{M}$ is a smooth, surjective map, known as the projection map. It is common to use the notation $E \xrightarrow{\pi} \mathcal{M}$ to denote a bundle.

Definition. [Fibre over $p \in \mathcal{M}$] Let $E \xrightarrow{\pi} \mathcal{M}$ be a bundle. We define the *fibre over* $p \in \mathcal{M}$ as the preimage under the projection, i.e.

$$\text{Fibre over } p := \text{preim}_\pi(p).$$

If the fibres to every point $p \in \mathcal{M}$ are all homeomorphic to the same topological space, say F, then we call the bundle a *fibre bundle with typical fibre* F.

Definition. [Trivial Bundle] A fibre bundle with typical fibre F, is called trivial if it is isomorphic to $\mathcal{M} \times F$.

Definition. [Section] Let $E \xrightarrow{\pi} \mathcal{M}$ be a fibre bundle.[15] Then a smooth map $\sigma : \mathcal{M} \to E$ is called a *section* if

$$(\pi \circ \sigma) = \mathbb{1}_{\mathcal{M}}.$$

Intuitively speaking, a section is a 'cut' of the total space; imagine passing a knife through it in one swoop, thereby picking out a single point in each fibre. However we can't do this randomly as we require the section to be smooth, which translates pictorially to the chosen points on neighbouring fibres to be 'close', i.e. we want to have a legitimate 'cut' of E rather then 'poking holes' in it. This is perhaps a subtle point, but it is important as it immediately tells us that not every bundle admits a *global* section, the typical example being a Möbius strip; once you go completely round the strip you do not end up where you started.[16] For this reason we sometimes talk about *local* sections. This idea will prove useful later.

[16] The only exception is if we go around the exact centre of the strip, but we get the point.

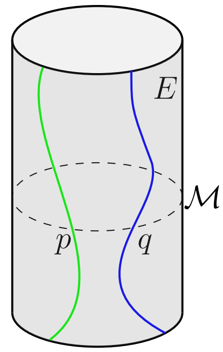


Figure 1.4: *Example of a trivial bundle and fibre. The total space, E, is the surface of the cylinder and the base space, $\mathcal{M}$, is the circle. The bundle is the triple consisting of E, $\mathcal{M}$ and a smooth, surjective projection map $\pi : E \to \mathcal{M}$. The preimage of the point p w.r.t. the projection map π is the green line — that is π maps every point on the green line to p — known as the fibre over p. Similarly the blue line is the fibre over q. The typical fibre is (homeomorphic to) $\mathbb{R}$. The section w.r.t. p, $\sigma_p : \mathcal{M} \to E$, maps p to a point within its fibre (a point on the green line). A map $\tau : \mathcal{M} \to E$ which maps p to a point in q's fibre (the blue line) is not a section, as $(\pi \circ \tau)(p) = q \neq \mathbb{1}_{\mathcal{M}}(p)$. The complete section is the set of points formed by taking one point from each fibre. The total space is clearly isomorphic to $S^1 \times \mathbb{R}$ and so it is a trivial bundle. Figure from my notes on [7].*

1.4.1 Tangent Bundle & Tensor Fields

An important example of a bundle is the tangent bundle. This is an example of a trivial bundle, where the typical fibres are the tangent planes, which are isomorphic to $\mathbb{R}^{\dim \mathcal{M}}$. These fibres are vector spaces, and so this vector space structure is inherited by the bundle itself. It is from here that we can begin to define the familiar notions of covariant derivative — see lectures 21-24 of [5] for more details.

The tangent bundle is useful for us as it allows us to get a precise definition of tensor fields. Namely, a smooth section of the tangent bundle gives us a vector *field*. This is not too hard to see: such a section is an allocation of a vector to each tangent space in a smooth fashion. This is *exactly* our intuitive notion of a vector field. We can easily define cotangent vector fields by defining the cotangent bundle analogously. From there we simply take tensor products to get our tensor *fields*.

Another important bundle for us is the so-called *frame bundle*. As the name suggests, the fibres here are the different frames we have. Noting that the set of frames is isomorphic to the Lorentz group (as the Lorentz group takes us from one frame to another), we can actually view the fibres as being $O(1,3)$. This is an example of what is known as a *principal G-bundle*, which is roughly described as a fibre bundle who's typical fibre is isomorphic to the group G. The tangent bundle is then a so-called *associated bundle* to the frame bundle. This basically just means that the principal G-bundle's group, for us $O(1,3)$, can act on the fibres of the associated bundle, for us the tensor fields. In this way we get a very precise definition of how the Lorentz group acts on our tensor fields. This construction is really important in a rigorous definition of *spinor* fields, however such a discussion would take us too long to include, and so the interested reader is directed towards lectures 19-20 of [5].

This might sound rather abstract — how do we imagine a group acting on the fibres? — but really it is much more familiar then we might think. Consider the case of the frame bundle and tangent bundle, as just described. Specifically, let's consider taking some fixed frame at $p \in \mathcal{M}$ and rotating it by some $R \in O(1,3)$ to get a new frame. In the principal G-bundle language, this is described by what is known as a *right* $O(1,3)$*-action*, and basically corresponds to moving within the fibre at p. This is just the statement that we are considering two different frames to p, related by the rotation R. Every frame at p can be viewed this way, i.e. given by the action of some $L \in O(1,3)$ on our given frame, and so we see the fibre at p is isomorphic to $O(1,3)$ itself. Now, we know from our experiences with GR that when we change frame, we should also change the components of our tensors, so that the abstract tensor itself remains unchanged. This is exactly how R acts on the associated bundle, the tangent bundle; it rotates the tensor in exactly such a way that the combined effort of rotating the frame and the tensor cancel. In the associated bundle language, the rotation of the tensor is given by a *left* $O(1,3)$*-action*. We have tried to depict this in Figure 1.5.

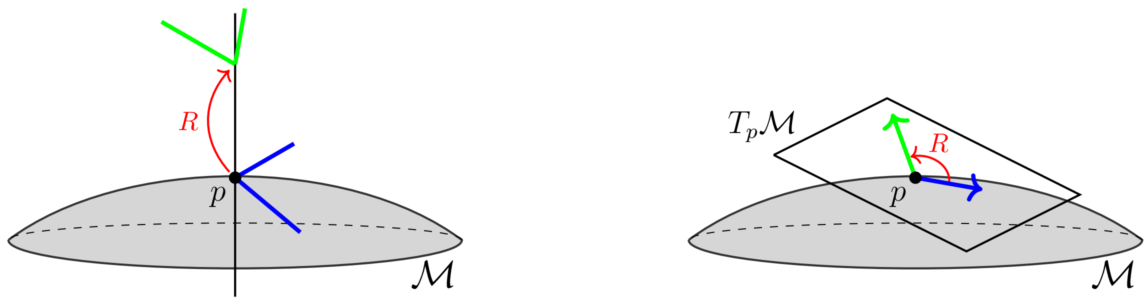


Figure 1.5: *Pictorial description of the action of a rotation $R \in O(1,3)$ to the fibres of: left) the frame (principal) bundle, and right) the tangent (associated) bundle. A fibre in the frame bundle (depicted by the vertical line) corresponds to all the different frames at $p \in \mathcal{M}$. For clarity, we have depicted the frames at the two points in the fibre; the green one being the rotated version of the initial blue one. This action in the frame bundle causes a related rotation in the tangent bundle, who's fibre at p is the 2-dimensional plane drawn. Again the blue arrow (representing the tensor) is rotated to the green one.*

2 | Lorentzian Spinors At A Point

Now that we have a good understanding of tensors and the Lorentz group, we can introduce *spinors* (at a point) on our Lorentzian manifold. The way we do this is not unique to twistor theory and plays vital roles in other areas, perhaps most notably in supersymmetry.

A lot of the material in this chapter, and the ones that follow, follows the information in [14], as well as the other references provided.

2.1 Double Cover Map

We start by considering a funny looking repackaging of the real, 4-dimensional vector $V^a = (V^0, V^1, V^2, V^3)^T$ via the following map:

$$\Psi(V^a) = V^{AA'} = \begin{pmatrix} V^{00'} & V^{01'} \\ V^{10'} & V^{11'} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} V^0 + V^3 & V^1 - iV^2 \\ V^1 + iV^2 & V^0 - V^3 \end{pmatrix} \tag{2.1}$$

where $A, A' \in \{1, 2\}$ and where the factor of $\frac{1}{\sqrt{2}}$ is a normalisation factor. This construction looks strange at first, but with a little work we can see that it just corresponds to sending

$$(V^0, V^1, V^2, V^3) \mapsto \frac{1}{\sqrt{2}}\big(\sigma_0 V^0, \sigma_1 V^1, \sigma_2 V^2, \sigma_3 V^3\big) \qquad \text{with,}$$

where $\sigma_0 = \mathbb{1}$ and σ_i are the Pauli matrices. In this way we construct a one-to-one mapping between 2×2 Hermitian matrices and elements of our vector space V. Why do we make such a construction? Well we now notice that the determinant is related to the spacetime length of V^a,

$$\det \Psi(V^a) = \frac{1}{2} \eta_{ab} V^a V^b, \tag{2.2}$$

and further if we multiply $V^{AA'}$ on the left by a $U \in SL(2, \mathbb{C})$ and on the right by the Hermitian conjugate $U^\dagger \in SL(2, \mathbb{C})$, the determinant is unchanged:

$$\det \widetilde{V}^{BB'} := \det \Big(U^A{}_B V^{AA'} (U^\dagger)_B^{B'} \Big) = \det V^{AA'}. \tag{2.3}$$

This procedure clearly also preserves the Hermiticity of $V^{AA'}$. As the determinant is unchanged, so to is the length, by Equation (2.2). So we have a linear transformation on vectors that leaves their length unchanged, but this is just a Lorentz transformation! As our mapping is linear, and that $\dim SL(2, \mathbb{C}) = 6 = \dim L$, this mapping is necessarily surjective, however it is clearly not *injective*. This is seen simply by the fact that if we had used $-U, -U^\dagger \in SL(2, \mathbb{C})$ in Equation (2.3) we would obtain exactly the same result. Another way to word this is that the kernel of our map, which we now denote $\rho : SL(2, \mathbb{C}) \to L$, is isomorphic to $\mathbb{Z}_2$,

$$\ker \rho := \{ U \in SL(2, \mathbb{C}) \,|\, \rho(U) = \mathbb{1}_{4\times 4} \} \cong \mathbb{Z}_2. \tag{2.4}$$

Now we note that, as $U \in SL(2, \mathbb{C})$ has $\det U = +1$, the map is onto L_+. Finally using that $SL(2, \mathbb{C})$ is simply connected[1] and that $L_+^\uparrow \cap L_+^\downarrow = \emptyset$, along with $\rho(\mathbb{1}_{2\times 2}) = \mathbb{1}_{4\times 4} \in L_+^\uparrow$, we see that

[1] Roughly speaking, a space is simply connected if any two points can be connected by a path, and any two such paths can be continuously deformed into each other.

$$\rho : SL(2,\mathbb{C}) \to L_+^\uparrow, \tag{2.5}$$

is a *2-to-1 isomorphism.*

Definition. [Spin Group] The (Lorentzian signature) spin group, $\text{Spin}(1, d-1)$, is defined to be the double covering of the orthochronus group $L_+^\uparrow$.

From the construction above and the definition just given, what we have just demonstrated is that

$$\text{Spin}(1,3) \cong SL(2,\mathbb{C}). \tag{2.6}$$

2.2 Spin-Space

We now return to Equation (2.1). Now note that if we are dealing with a null-vector, $g(V,V)=0$, then by Equation (2.2) our matrix is degenerate. This drops the rank of our matrix from two to one, and thus we may write $V^{AA'}$ in terms of the outer product between two complex 2-dimension vectors, related by complex conjugation:

$$V^{AA'} = \begin{pmatrix} \alpha^0\bar{\alpha}^{0'} & \alpha^0\bar{\alpha}^{1'} \\ \alpha^1\bar{\alpha}^{0'} & \alpha^1\bar{\alpha}^{1'} \end{pmatrix} = \alpha^A\bar{\alpha}^{A'}, \tag{2.7}$$

where we have left the $\otimes$ symbol implicit on the right-hand side. So in total we have reduced the consideration of the space of 4-dimensional real null vectors to the consideration of a complex 2-dimensional vector space, $\mathbb{S}$, and its complex conjugate, $\bar{\mathbb{S}} = \mathbb{S}'$. This vector space $\mathbb{S}$ is known as *spin-space*, and the elements $\alpha^A \in \mathbb{S}$ and $\beta^{A'} \in \mathbb{S}'$ are our *spinors*. The elements in Equation (2.7) are related by Hermitian conjugation

$$\bar{\alpha}^{A'} = \big(\alpha^A\big)^\dagger. \tag{2.8}$$

Just as we had the covector space dual to V, we also have the dual spin-spaces $\mathbb{S}^*$ and $\mathbb{S}'^*$, with elements $\alpha_A \in \mathbb{S}^*$ and $\beta_{A'} \in \mathbb{S}'^*$. Again we can relate elements in these two spaces by Hermitian conjugation

$$\bar{\alpha}_{A'} = \big(\alpha_A\big)^\dagger.$$

2.2.1 Higher Valence Spinors

As we tried to stress above when constructing our tensors, the constructions apply to *any* vector space, not just the tangent vector spaces. We can therefore immediately import these ideas in order to construct higher valence spinors, however we now need to keep track of the fact that we have both primed and unprimmed indices. So, for example, we can have

$$\alpha^{AB'C'}{}_{DE'} \in \mathbb{S}\otimes\mathbb{S}'\otimes\mathbb{S}'\otimes\mathbb{S}^*\otimes\mathbb{S}'^*.$$

Hermitian conjugation then just extends from the explanations above, so that,

$$\big(\alpha^{AB'C'}{}_{DE'}\big)^\dagger = \bar{\alpha}^{A'BC}{}_{D'E} \in \mathbb{S}'\otimes\mathbb{S}\otimes\mathbb{S}\otimes\mathbb{S}'^*\otimes\mathbb{S}^*.$$

It follows from this that we can only have a Hermitian spinor if we have the same number of primed and unprimmed spinors in the same raised/lowered way, i.e. we need the same number of copies of $\mathbb{S}$ and $\mathbb{S}'$ and the same number of copies of $\mathbb{S}^*$ and $\mathbb{S}'^*$. For example, $\alpha^{ABC'D'}$ is Hermitian if

$$\bar{\alpha}^{A'B'CD} = \alpha^{A'B'CD} \qquad \stackrel{\text{e.g.}}{\Longrightarrow} \qquad \bar{\alpha}^{0'0'01} = \big(\alpha^{000'1'}\big)^\dagger = \alpha^{000'1'}.$$

From here we can show that there is a 1-1 correspondence between real (r,s)-tensors and Hermitian spinors with r $\mathbb{S},\mathbb{S}'$ indices and s $\mathbb{S}^*,\mathbb{S}'^*$ indices [14]. The particular case of a $(1,0)$-tensor, i.e. a vector, is simply Equation (2.7); V is in the Hermitian part of $\mathbb{S}\otimes\mathbb{S}'$.

It is in this way that people often say that spinors are more elementary then tensors: given any real tensor we can represent it in spinor language, however the reverse is not true; we cannot write a non-Hermitian spinor as a vector. This is true, however we should be a bit careful; as we explained before, the construction of a tensor field is purely topological, i.e. we don't need a metric structure. This, however, is *not* true for spinors: in order to define a spinor field we *must* equip our manifold with some form of metric. A proper understanding of this comes from a principal G-bundle formalism: the idea is that we need to define the *spin frame bundle*, which is obtained from the frame bundle mentioned previously along with our double covering map. However this map is onto $L_+^\uparrow$, and so in order to construct the spin frame bundle we *need* a metric space. Spinor fields are then sections of the appropriate associated bundle, called *spin-bundles*. See [5, 8] for more details.

2.2.2 $SO(4,\mathbb{C})$

In twistor theory it is often useful to consider complexified spacetime, in which case our Lorentz group becomes $SO(4,\mathbb{C})$. The reason this is important is it gives us another interesting isomorphism that works well with Equation (1.9) [11]:

The group $SO(4,\mathbb{C})$ is not simple and decomposes as

$$SO(4,\mathbb{C})\cong\big(SL(2,\mathbb{C})\times\widetilde{SL}(2,\mathbb{C})\big)/\mathbb{Z}_2. \tag{2.9}$$

The two spin-bundles, $\mathbb{S}$ and $\mathbb{S}'$, are representation spaces of $SL(2,\mathbb{C})$ and $\widetilde{SL}(2,\mathbb{C})$, respectively.

The reason we consider complexified spacetime is that it encompasses every real spacetime signature. That is, we formulate our constructions in this complexified spacetime and then at the end of the calculation take a specific real slice to give us our desired signature. We summarise the three slices here [11]

- Lorentzian: We have just seen that in Lorentzian signature we have $\text{Spin}(1,3)\cong SL(2,\mathbb{C})$. We have also just highlighted that in Lorentzian signature, the two copies of $SL(2,\mathbb{C})$ in Equation (2.9) are related by complex conjugation.
- Riemannian: In this signature we have $\text{Spin}(4,0)\cong SU(2)\times\widetilde{SU}(2)$.
- Split: Here we get $\text{Spin}(2,2)\cong SL(2,\mathbb{R})\times\widetilde{SL}(2,\mathbb{R})$. It is only in this signature that we have a notion of real spinors.

2.2.3 Raising & Lowering

Of course we can also inherit the (anti)symmetrisation procedures to our spinor tensors. However we note that, as our spin space is 2-dimensional, our top forms are 2-forms. In other words, up to complex multiples, there is a unique (non-zero) 2 index spinor form. As before, our reference choice is the Levi-Civita symbol, ϵ_{AB}. We also have a 2-form on our complex conjugate space, which is simply given by $\bar{\epsilon}_{A'B'}\equiv\epsilon_{A'B'}$, where the right-hand side is a notational brevity we introduce. We also have the raised index versions ϵ^{AB} and $\epsilon^{A'B'}$. These are defined via the following contractions

$$\epsilon^{AB}\epsilon_{CB}=\delta_C{}^B, \tag{2.10}$$

where care must be taken to note the placement of indices, as these are antisymmetric. We can raise and lower our spinor indices using these simply as

$$\begin{aligned}\epsilon_{AB}&:\mathbb{S}\to\mathbb{S}^*\\ \alpha^A&\mapsto\alpha_A:=\alpha^B\epsilon_{BA}\end{aligned}$$

and

$$\begin{aligned}\epsilon^{AB} &: \mathbb{S}^* \to \mathbb{S} \\ \alpha_A &\mapsto \alpha^A := \epsilon^{AB}\alpha_B,\end{aligned}$$

Again care must be taken with the ordering of indices: for unprimed indices we contract 'top left to bottom right'. Of course the same ideas apply for the primed indices, but now the contraction order to changed to 'bottom left to top right', i.e.

$$\begin{aligned}\epsilon_{A'B'} &: \mathbb{S}' \to \mathbb{S}'^* \\ \alpha^{A'} &\mapsto \alpha_{A'} := \epsilon_{A'B'}\alpha^{B'},\end{aligned}$$

and similarly for $\epsilon^{A'B'}$.

ϵ_{AB} and $\epsilon_{A'B'}$ are actually examples of what are known as *symplectic forms*. This just means they are closed (as they are top-forms), non-degenerate, 2-forms. We then say that $\mathbb{S}$ and $\mathbb{S}'$ are *symplectic manifolds*. We will not use this language much, but this comment is just included to make comparison to references, e.g. [11], easier.

Next we note that, for any $U \in SL(2,\mathbb{C})$

$$U_A{}^B U_C{}^D \epsilon_{BD} = \det U \epsilon_{AC} = \epsilon_{AC},$$

which is the spin space equivalent of Equation (1.13). Putting all this together suggests that there is some sort of link between the spacetime metric[2] η_{ab} and our 2-forms. Recalling that a real $(0,2)$-tensor is in 1-to-1 correspondance with a Hermitian spinor with no $\mathbb{S}/\mathbb{S}'$ indices and 2 $\mathbb{S}^*$ and $\mathbb{S}'^*$ indices, we propose the ansatz

$$\eta_{ab} \sim \epsilon_{AB}\epsilon_{A'B'}.$$

Claim 2.2.1. If we choose our ϵ_{AB} such that $\epsilon_{01} = 1$ — which implies $\epsilon_{0'1'} = 1$ by Hermitian conjugation — then the exact correspondance is

$$\eta_{ab} = \epsilon_{AB}\epsilon_{A'B'}. \tag{2.11}$$

Proof. We have already argued that the right-hand side of Equation (2.11) corresponds to some real $(0,2)$-tensor. Next we note that it is symmetric as

$$\eta_{ba} = \epsilon_{BA}\epsilon_{B'A'} = (-1)^2\epsilon_{AB}\epsilon_{A'B'} = \eta_{ab}.$$

Next note that all spinors are null w.r.t. $\epsilon_{AB}/\epsilon_{A'B'}$ simply by antisymmetry

$$\epsilon_{AB}\alpha^A\alpha^B = \epsilon_{[AB]}\alpha^{(A}\alpha^{B)} = 0,$$

where we have used that $\alpha^A\alpha^B = \alpha^{(A}\alpha^{B)}$ and the general tensor rule $T_{[AB]}S^{(AB)} = 0$. Putting this together with Equation (2.7), i.e. that any null vector is of the form $V^a = \alpha^A\bar{\alpha}^{A'}$, we have

$$\eta_{ab}V^aV^b = \epsilon_{AB}\epsilon_{A'B'}\alpha^A\bar{\alpha}^{A'}\alpha^B\bar{\alpha}^{B'} = \big(\epsilon_{AB}\alpha^A\alpha^B\big)\big(\epsilon_{A'B'}\bar{\alpha}^{A'}\bar{\alpha}^{B'}\big) = 0,$$

and so we are correct up to a normalisation constant. Finally we fix this by considering a unit time-like vector[3] $T^a \mapsto t^{AA'}$, i.e.

$$\Psi(T^a) = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},$$

then

$$\eta_{ab}T^aT^b = \epsilon_{AB}\epsilon_{A'B'}t^{AA'}t^{BB'} = \epsilon_{01}\epsilon_{0'1'}t^{00'}t^{11'} + \epsilon_{10}\epsilon_{1'0'}t^{11'}t^{00'} = 1,$$

which is the required normalisation. □

Equation (2.11) is another important result, and we rewrite it here in a slightly nicer notation for what is to follow.

[2]From now on, unless otherwise specified, we shall work in Minkowski spacetime, so $g \to \eta$. A justifcation for this shall be given later.

[3]Note that we do *not* have a decomposition into $t^A\bar{t}^{A'}$, as T^a is not null.

If α, β are spinor fields in $\mathbb{S}$ and α', β' are spinor fields in $\mathbb{S}'$, then we have a relation

$$\eta\big(\alpha \otimes \alpha', \beta \otimes \beta'\big) = \epsilon\big(\alpha, \beta\big)\epsilon'\big(\alpha', \beta'\big). \tag{2.12}$$

It is the interplay between Equations (1.9), (2.9) and (2.12) that makes twistor theory particularly effective in 4-dimensions [11].

2.3 Spinor Decomposition & Principal Null Directions

2.3.1 Dyads

We are now in a position to observe some nice properties of higher rank spinors. First we note that, just as a point in our tangent space is given by a *tetrad* (i.e. a basis), a point in our spin space $\mathbb{S}$ is given by a *dyad*, which we denote (o_A, ι_A). We normalise this dyad by imposing

$$\epsilon_{AB} o^A \iota^B = o_B \iota^B = 1. \tag{2.13}$$

As our spinors provide a double cover of $L_+^\uparrow$, we can use our dyad to define a tetrad. We do this by defining[4]

$$\ell^a := o^A \bar{o}^{A'}, \qquad n^a := \iota^A \bar{\iota}^{A'}, \qquad m^a := o^A \bar{\iota}^{A'}, \quad \text{and} \quad \bar{m}^a := \iota^A \bar{o}^{A'}. \tag{2.14}$$

Using Equation (2.11), we can readily verify that these are all null. We also note that ℓ^a and n^a are *real*, as their spinor decomposition is Hermitian. However m^a and $\bar{m}^a$ are *complex*, and are related by Hermitian conjugation. It is also easy to check that the only non-vanishing inner-products are

$$\eta_{ab}\ell^a n^b = 1 = -\eta_{ab} m^a \bar{m}^b. \tag{2.15}$$

On top of this, the tetrad is also right-handed. Such a tetrad is referred to as a *null tetrad*.

As our dyad forms a basis for $\mathbb{S}$, we should be able to express our 2-form in terms of a normalised dyad (o_A, ι_A). Using antisymmetry and the fact that we want $\epsilon_{01} = 1$ (so that Equation (2.11) holds), we are lead to conclude that

$$\epsilon_{AB} = o_A \iota_B - \iota_A o_B. \tag{2.16}$$

The reason we're doing all this is to prove the following claim.

<u>*Claim 2.3.1*</u>. The following contraction holds

$$\epsilon_{AB}\epsilon^{CD} = \delta_A{}^C \delta_B{}^D - \delta_A{}^D \delta_B{}^C. \tag{2.17}$$

Proof. We show this by considering each case in turn.

- First we note that if $A = B$ *and/or* $C = D$ then, by antisymmetry, the expression vanishes.
- Now consider $A = C$ and $B = D$, then Equation (2.16) gives us

$$\epsilon_{AB}\epsilon^{CD}\delta_C{}^A\delta_B{}^D = \big(o_A\iota_B - \iota_A o_B\big)\big(o^A\iota^B - \iota^A o^B\big)\delta_C{}^A\delta_B{}^D.$$

  Now use Equation (2.13) and the null conditions

$$o_A o^A = 0 = \iota_A \iota^A,$$

  so that we are just left with

$$\epsilon_{AB}\epsilon^{AB} = -o_A\iota^A\iota_B o^B = +o_A\iota^A o_B \iota^B = 1,$$

  where we have used $\iota_B o^B = \iota^A \epsilon_{AB} o^B = -\iota^C \epsilon_{BA} o^B = -\iota^A o_A$.
- The case for $A = D$ and $B = C$ follows similarly to the above but we get a minus sign.

∎

[4]Note that our double cover is seen from the fact that $(o_A, \iota_A) \to (-o_A, -\iota_A)$ gives the same result.

2.3.2 Flagpoles & Flagplanes

We have constructed our dyads (o_A, ι_A) and shown how we can relate them to a tetrad for our spacetime. We can now give a nice geometrical picture for these dyads, in particular in relation to Minkowski spacetime, Figure 2.1b.

The first thing we note is that, as we have seen, any spinor o^A defines a real null vector $\ell^a = o^A \bar{o}^{A'}$. We call such a vector the *flagpole* of o^A. Next we claim that any non-zero spinor o^A defines a real, simple null bivector (here written as a 2-form) via [15]

$$P_{ab} = o_A o_B \epsilon_{A'B'} + \bar{o}_{A'} \bar{o}_{B'} \epsilon_{AB}.$$

If we then complete the dyad by introducing a spinor ι_A, we can use Equation (2.16) (and the barred version) to write P_{ab} as

$$\begin{aligned} P_{ab} &= o_A o_B \big(\bar{o}_{A'} \bar{\iota}_{B'} - \bar{\iota}_{A'} \bar{o}_{B'}\big) + \bar{o}_{A'} \bar{o}_{B'} \big(o_A \iota_B - \iota_A o_B\big) \\ &= \ell_a m_b - \ell_b m_a + \ell_a \bar{m}_b - \ell_b \bar{m}_a \\ &= 2\ell_{[a} X_{b]}, \end{aligned}$$

where we have used Equation (2.14) and defined

$$X_a := m_a + \bar{m}_a.$$

The last line above also demonstrates explicitly that P_{ab} is simple.

If we now again recall the inner products for our tetrad, Equation (2.15), we see that X_a is orthogonal to ℓ_a and is spacelike. Finally we note that the choice of ι_A was not unique, and we could just have easily used $\widetilde{\iota}_A = \iota_A + \lambda o_A$, as this would still satisfy Equation (2.13) by $o_A o^A = 0$. Again recalling Equation (2.14), we see that such a change in ι_a would result in

$$m_a = \iota_A \bar{o}_{A'} \mapsto \widetilde{m}_a = (\iota_A + \lambda o_A)\bar{o}_{A'} = m_a + \lambda \ell_a,$$

and similarly $\bar{m}_a \mapsto \bar{m}_a + \lambda \ell_a$. So we see that X_a is altered by a factor of ℓ_a.

In this way we see that o_A not only defines our flagpole ℓ_a, but it also defines a null[5] 2-plane (given by shifting X_a along ℓ_a) 'anchored' around the flagpole ℓ_a. We call this plane a *flagplane*, and the collection of the flagpole and flagplane the *null flag*. Hopefully the geometrical picture, Figure 2.1a, helps with understanding the names.

2.3.3 Principal Null Directions

We now want to draw the following, somewhat unexpected, conclusion: only symmetric spinors matter. It is not immediately clear why this is true, so let's flush it out. Suppose we have some generic (r, s)-spinor that is antisymmetrised on two indices. W.l.o.g. take these to be $\mathbb{S}^*$ indices, $\Phi_{...CD...} = \Phi_{...[CD]...}$, where the "..."s indicate other indices of any kind. Now multiplying by Equation (2.17), we have

$$\epsilon_{AB} \epsilon^{CD} \Phi_{...CD...} = 2\Phi_{...AB...},$$

but we equally could have used the ϵ^{CD} on the left-hand side to raise the D index

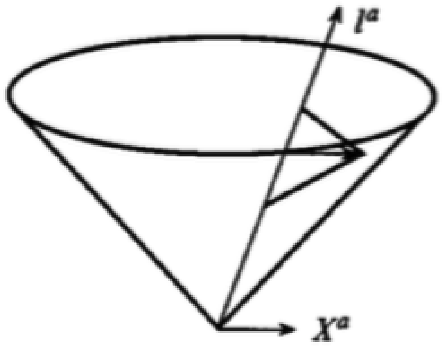
$$\epsilon^{CD} \Phi_{...CD...} = \Phi_{...C}{}^{C}{}_{...},$$

in other words we can replace the antisymmetrised indices with a ϵ_{AB}:

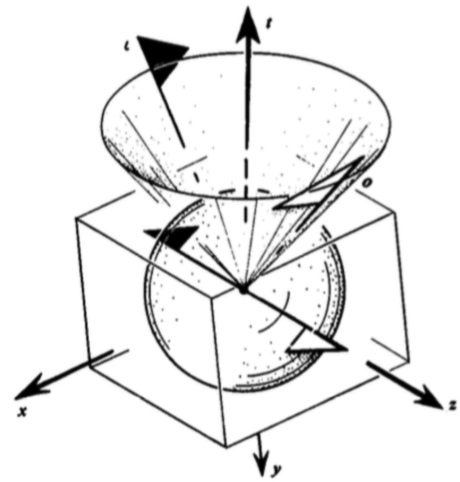
$$\Phi_{...AB...} = \frac{1}{2} \epsilon_{AB} \Phi_{...C}{}^{C}{}_{...}. \tag{2.18}$$

Of course a similar argument follows for antisymmetrised $\mathbb{S}, \mathbb{S}'$ and $\mathbb{S}'^*$ indices, with the respective ϵ/ϵ's used. This is a very neat result and so we stress it again: *any* spinor can be written in terms of symmetric spinors and the ϵ/ϵ's. This simplifies the problem greatly.

[5] In the sense that a 2-plane in Minkowski spacetime is null iff it contains at least one null vector, here ℓ^a, and all other vectors are null or spacelike. We move away from the flagpole with X^a, which is spacelike.



(a) *Pictorial depiction of a flagpole and flagplane. Image from [14].*



(b) *Pictorial depiction of the relation of a normalised dyad (o_A, ι_A), with the resulting null flags, and the standard normalised tetrad (t, x, y, z) of Minkowski spacetime. Image from [15].*

Figure 2.1: *Pictorial depictions of null flags.*

However, there is, in fact, a further simplification that comes from the fact that our spin spaces are complex. Consider a general $(0, n)$-spinor $\Phi_{A_1...A_n}$. Now contract every index with $\xi^{A_i} = (1, x)$, then the result $\xi^{A_1}...\xi^{A_n}\Phi_{A_1...A_n}$ is a polynomial of degree n in x. Finally, as our space is complex, the solutions are always valid, and so $\Phi_{A_1...A_n}$ factorises. This tells us that we can write $\Phi_{A_1...A_n}$ as the symmetrised outer product over n spinors,

$$\Phi_{A_1...A_n} = \alpha_{(A_1}...\beta_{A_n)}. \tag{2.19}$$

The individual spinors make up the linear factors (i.e. $\xi^2\xi^1...\xi^1\Phi_{21...1} = (\alpha_1 + ... + \beta_1)x)$, and are, of course, only defined up to a scale. From Equation (2.7), we can construct a real null vector for each of these spinors, and these null vectors define the *principal null directions* (p.n.ds) of $\Phi_{A_1...A_n}$.

2.3.4 2-Form Decomposition

As we have mentioned a few times, there is a particular type of real vector we are interested in, a bivector. In fact we are interested in the lowered index version, the 2-forms. In order to find their spinor decomposition, let's consider a general (i.e. not antisymmetrised) $(0, 2)$-vector, T_{ab}. As it is a real vector, we know that we can write it in terms of a Hermitian spinor with 2 $\mathbb{S}^*$ and $\mathbb{S}'^*$ indices,[6]

$$T_{ab} = T_{ABA'B'}.$$

Now use the fact that a general 2 index object can be written as the sum of the symmetric and antisymmetric part to rewrite this as

$$\begin{aligned} T_{ABA'B'} &= T_{(AB)A'B'} + T_{[AB]A'B'} \\ &= T_{(AB)A'B'} + \frac{1}{2}\epsilon_{AB}T_C{}^C{}_{A'B'}, \end{aligned}$$

where we have made use of Equation (2.18). We can then do a similar thing on the $A'B'$ indices and obtain four total terms,

$$T_{ab} = T_{(AB)(A'B')} + \frac{1}{2}\Big(\epsilon_{AB}T_C{}^C{}_{A'B'} + \epsilon_{A'B'}T_{(AB)C'}{}^{C'}\Big) + \frac{1}{4}\epsilon_{AB}\epsilon_{A'B'}T_{CC'}{}^{CC'}$$

We can break these terms down in turn:

[6] Here we have used our ordering isomorphisms to place all unprimed indices first.

- $T_{(AB)(A'B')}$ is fully symmetrised across the $\mathbb{S}^*$ and $\mathbb{S}'^*$ indices and so decomposes exactly into its p.n.d.s and defines a Hermitian spinor corresponding to the trace-free part of T_{ab}.
- The reason we had the trace-free part above is because $\frac{1}{4}\epsilon_{AB}\epsilon_{A'B'}T_{CC'}{}^{CC'} = \frac{1}{4}\eta_{ab}T_c{}^c$, via Equation (2.11), and so corresponds to the trace of T_{ab}.
- The two terms grouped in the parentheses are Hermitian conjugate of each other, and so collectively define a real tensor. Putting this together with the fact that interchanging $A \leftrightarrow B$ and $A' \leftrightarrow B'$ gives an overall minus sign, we see that this corresponds to our 2-form, $T_{[ab]}$.

So in total we conclude that a real 2-form F_{ab} decomposes in spin space as

$$\begin{aligned} F_{ab} &= \Phi_{AB}\epsilon_{A'B'} + \bar{\Phi}_{A'B'}\epsilon_{AB} \\ &= \alpha_{(A}\beta_{B)}\epsilon_{A'B'} + \bar{\alpha}_{(A'}\bar{\beta}_{\beta')}\epsilon_{AB} \end{aligned} \tag{2.20}$$

where $\Phi_{AB} = \Phi_{(AB)}$ with p.n.d.s α/β.

We can investigate the effect of the Hodge dual by recalling Equation (1.7), along with[14]

$$\epsilon_{abcd} = i\big(\epsilon_{AC}\epsilon_{BD}\epsilon_{A'D'}\epsilon_{B'C'} - \epsilon_{AD}\epsilon_{BC}\epsilon_{A'C'}\epsilon_{B'D'}\big)$$

so that

$$\epsilon_{ab}{}^{cd} = i\big(\delta_A{}^C\delta_B{}^D\delta_{A'}{}^{D'}\delta_{B'}{}^{C'} - \delta_A{}^D\delta_B{}^C\delta_{A'}{}^{C'}\delta_{B'}{}^{D'}\big).$$

That is,[7]

$$(\star F)_{ab} = \frac{1}{2}\epsilon_{ab}{}^{cd}F_{cd} = -i\Phi_{AB}\epsilon_{A'B'} + i\bar{\Phi}_{A'B'}\epsilon_{AB}.$$

This allows us to clearly see the decomposition into self-dual and anti-self-dual, Equation (1.9): the space of self-dual tensors are those 2-forms with spinor decomposition

$$W^+_{ab} = \bar{\Phi}_{A'B'}\epsilon_{AB},$$

while the space of anti-self-dual 2-forms are those with

$$W^-_{ab} = \Phi_{AB}\epsilon_{A'B'}. \tag{2.21}$$

From here we can further see the interplay between our three important isomorphisms, Equations (1.9), (2.9) and (2.12): the first two are related by [11]

$$\Lambda^2_+\mathcal{M} \cong \mathbb{S}'^* \odot \mathbb{S}'^* \quad \text{and} \quad \Lambda^2_-\mathcal{M} \cong \mathbb{S}^* \odot \mathbb{S}^*, \tag{2.22}$$

where $\odot$ is the symmetric tensor product,

$$A \odot B := A \otimes B + B \otimes A.$$

So far we have demonstrated that a lot of physics can be viewed as a consequence of the 2-spinor formalism. That is, it follows from this approach that we want our spacetime to be 4-dimensional, and on top of this we require such a spacetime to be globally orientable, both in terms of handedness and time orientability, as these are needed to define $L^\uparrow_+$, which in turn is needed to define a spinor field. We could therefore take the view that, rather than simply giving them, these concepts are *derived* from the 2-spinor algebra.

This is a very powerful statement, but we could still do better. Up until now, we have had to provide our manifold $\mathcal{M}$ by hand, given that it is somewhat restricted by the above requirements. If we could somehow also view the points in $\mathcal{M}$ as a derived consequence of our spinors, we would then be in a very strong position indeed. This is where twistor theory comes in, with the ultimate goal of "reformulating the whole of basic physics in twistor terms" [15]. To word it another way, just as we can view spinors as more primitive than vectors, we can chose to view "twistor algebra as more primitive than spacetime itself."

[7] We are working with Minkowski spacetime so the metric determinant factor is just 1.

3 | Twistor Space

We begin our study of twistor space by introducing the two main field equations, and then proceed to give a definition of twistor space in terms of projective spaces. This will then let us examine the link between twistor space and spacetime through the twistor correspondence.

3.1 Zero Rest Mass & Twistor Equations

We are familiar with connections from our experiences in GR, however there they are often just introduced as an operator on tensors with given properties. As we mentioned before, the full construction of connections come from considering principal G-bundles, and the interested reader is directed to lectures 21-25 of [5].

Ok so given that we have a Levi-Civita connection on our frame bundle over $\mathcal{M}$, ∇_a, we recall again that we can reconstruct any real vector from spinors. In particular, Equation (2.21) tells us that a spinor field defines a null, anti-self-dual 2-form (up to sign). We therefore can extend our Levi-Civita connection uniquely to define one on our spin bundles. In order to meet the metric compatibility condition, we require

$$\nabla_a \epsilon_{BC} = 0 = \nabla_a \epsilon_{B'C'}.$$

3.1.1 Zero Rest Mass Equations

We can now start to construct spinor field equations, i.e. the spinor equivalent of things such as Maxwell's field equations,

$$\nabla_{[a} F_{bc]} = 0 \quad \text{and} \quad \nabla^a F_{ab} = 0,$$

where $F \in \Lambda^2\mathcal{M}$ is the Maxwell field strength tensor. Indeed this is where we start. F is a 2-form, so we can decompose it into its self-dual and anti-self-dual parts. With the above extension in mind, we consider the anti-self-dual part,

$$W_{ab} := F_{ab} + i(\star F)_{ab}.$$

We can then write Maxwell's equations compactly as the single equation[1]

$$\nabla^a W_{ab} = 0.$$

This is all in form language. We convert it into spinors by recalling Equation (2.21), $W_{ab} = \Phi_{AB}\epsilon_{A'B'}$, to give us

$$\nabla_{A'}{}^{A} \Phi_{AB} = 0. \tag{3.1}$$

This is an example of a more general set of spinor field equations, known as *zero rest mass free field equations*, or just z.r.m equations. In order to "see" the extension, we give another example: a linearised solution of Einstein's vacuum field equations in Minkowski space is given by the z.r.m equation

$$\nabla_{A'}{}^{A} \Phi_{ABCD} = 0,$$

where Φ_{ABCD} is totally symmetric [14]. If we then recall that Maxwell's equations correspond to photons and Einstein's equations to gravitons, which have helicities $|s| = 1$ and $|s| = 2$, respectively, we argue the extension is thus.[2]

[1] In exterior derivative language, Maxwell's equations read $dF = 0$ and $d \star F = 0$. The contraction with ∇^a corresponds to d, and so we see how this equation contains both.

[2] We change notation, $\Phi \to \varphi$, from here on, in order to make comparisons with [14] easier.

A symmetric spinor field, corresponding to a field with helicity $-s$, is a symmetric, $2s$ index spinor $\varphi_{A\ldots B}$ that obeys the z.r.m equation

$$\nabla_{A'}{}^{A}\varphi_{A\ldots B} = 0. \tag{3.2}$$

If the helicity is $+s$, then we have primed spinor indices instead

$$\nabla_{A}{}^{A'}\varphi_{A'\ldots B'} = 0. \tag{3.3}$$

We will study the solutions to Equations (3.2) and (3.3) in much more detail later.

3.1.2 Twistor Equation

The other important equation we need is the so-called *twistor equation*. It is simply given by

$$\nabla_{A'}{}^{(A}\omega^{B\ldots C)} = 0, \tag{3.4}$$

where $\omega^{B\ldots C}$ is a totally symmetric spinor field.

The first important thing we notice is that if the spinor field Ω^B satisfies the twistor equation, i.e.

$$\nabla_{A'}{}^{(A}\Omega^{B)} = 0,$$

then, by considering symmetry and index structure arguments,

$$\nabla_{A'}{}^{A}\Omega^{B} = -i\epsilon^{AB}\pi_{A'} \tag{3.5}$$

holds, where $\pi_{A'}$ is some other spinor field. The factor of $-i$ is typically only introduced when considering Lorentzian signature Minkowski spacetime [16]. However here we shall motivate it in a general sense simply by a redefinition of $\pi_{A'}$, and then see in Section 3.3.4 the role this i factor plays for the specific case of Lorentzian signature.

From here we make the claim — see Chapter 7 of [14] — that, when our spacetime is (conformally)[3] flat, that the general solution to Equation (3.5) is

$$\Omega^{A}(x) = \omega^{A} - ix^{AA'}\pi_{A'}, \tag{3.6}$$

where ω^A is a constant spinor field, given by the value of $\Omega^A(x)$ at the origin $x^{AA'} = 0$.[4] We call such a spinor[5] $\Omega^A(x)$ a *twistor* and the vector space of solutions to Equation (3.6) is our *twistor space*, denoted $\mathbb{T}$.

We take the $x^{AA'} \in \mathbb{C}$, so that we are considering complex Minkowski spacetime $\mathbb{M}_{\mathbb{C}}$, quoting the motto "complexify first, ask questions later"[16]. We therefore see that twistor space corresponds to a 4-complex-dimensional vector space, which we coordinatise with the standard notation $(Z^0, Z^1, Z^2, Z^3) \in \mathbb{C}^4$. We can write these coordinates in terms of a choice of origin (so that $x^{AA'}$ is defined) and by a pair of spinors of opposite chirality, i.e.[6]

$$Z^{\alpha} = (\omega^{A}, \pi_{A'}), \qquad \alpha = 0,1,2,3, \quad \text{and} \quad A, A' = 1,2. \tag{3.7}$$

To be clear, we replace *all* 4 coordinates (Z^0, Z^1, Z^2, Z^3) with $(\omega^A, \pi_{A'})$. From here, and Equation (3.6), we may therefore refer to Z^α as the twistor. That is Z^α defines a point in twistor space, and therefore defines a twistor $\Omega^A(x)$.

[3] This just means up to boundary conditions at infinity.

[4] Other notation exists, in particular [15] uses a circle above a quantity to indicate its value at the origin. That is Equation (3.6) is written as $\omega^A = \mathring{\omega}^A - ix^{AA'}\pi_{A'}$ and $\pi_{A'} = \mathring{\pi}_{A'}$ there.

[5] Henceforth we shall say "spinor" to mean "spinor field" just to save space.

[6] The reason we express Z^α in terms of both ω^A and $\pi_{A'}$, rather then just with Ω^A is that it makes the algebra of higher-valence twistors easier to manipulate [15].

Of course we also have the conjugate and dual twistor spaces, and so we initially expect to have lowered and primed twistor indices. Indeed we do, however we now note that we can define a pseudo-Hermitian inner product on twistor space via [14]

$$\Sigma(Z) \equiv Z \cdot \bar{Z} \equiv \Sigma_{\alpha\beta'} Z^\alpha \bar{Z}^{\beta'} := \omega^A \bar{\pi}_A + \bar{\omega}^{A'} \pi_{A'}, \tag{3.8}$$

where $\bar{Z}^{\beta'} := (Z^\beta)^\dagger \in \bar{\mathbb{T}}$ is a conjugate twistor. However, this inner product is non-degenerate, and so it allows us to identify $\bar{\mathbb{T}}$ with $\mathbb{T}^*$, the dual space to $\mathbb{T}$. In this way, we can forget about primed twistor indices altogether. In other words, we can view $\dagger$ as a map from $\mathbb{T}$ to the dual space $\mathbb{T}^*$ via

$$Z^\alpha = (\omega^A, \pi_{A'}) \mapsto \bar{Z}_\alpha = (\bar{\pi}_A, \bar{\omega}^{A'}). \tag{3.9}$$

We call $\bar{Z}_\alpha \in \bar{\mathbb{T}}$ a *dual twistor* and $\bar{\mathbb{T}}$ is our dual twistor space. It is important to note that it is the *twistor* indices, i.e. α, β etc, which do not appear primed; of course the spinor indices, A, B etc, still appear primed in Equation (3.9). We can remember how this works with the following mantra "to conjugate a twistor, we conjugate all its spinor parts, and then place each conjugated part into the correct position, namely that appropriate for a twistor with all original twistor indices at reversed level"[15].

We can use this inner product to split $\mathbb{T}$ into three parts:

(i) $\Sigma(Z) > 0$ gives $\mathbb{T}^+$,

(ii) $\Sigma(Z) < 0$ gives $\mathbb{T}^-$, and

(iii) $\Sigma(Z) = 0$ gives $\mathbb{N}$.[7]

We will return to these spaces shortly.

3.1.3 Incidence Relations

Before moving on to discuss projective spaces, we introduce "the root of everything interesting about twistor theory"[16], the *incidence relations*.

Consider some twistor $\Omega^A(x)$, and consider the points in $\mathbb{M}_\mathbb{C}$ where such a twistor vanishes. From Equation (3.6), that is consider the points such that

$$\boxed{\omega^A = i x^{AA'} \pi_{A'}.} \tag{3.10}$$

We now see why we consider complex Minkowski spacetime: it's Minkowski as we needed our spacetime to be (conformally) flat in order to arrive at Equation (3.6), and we require $x^{AA'} \in \mathbb{C}$ for solutions to Equation (3.10) to exist in general. The incidence relation gives us a relationship between points in twistor space and points in $\mathbb{M}_\mathbb{C}$. It is from here that we can begin to view spacetime points as derived concepts from the twistor algebra. We will explore the geometry of this relationship in more detail after introducing projective twistor space.

Note that Equation (3.10) is linear and holomorphic, in the sense that no complex conjugated spinors appear in it. This will prove important in a moment.

We similarly get an incidence relation for dual twistors by taking the dual of Equation (3.10). As we just explained, this corresponds to just taking the complex conjugation of Equation (3.10), i.e.[8]

$$\bar{\omega}^{A'} = -i (x^{AA'})^\dagger \bar{\pi}_A.$$

We now get a hint at the simplification of including a factor of i in Equation (3.10): it gives us a minus sign here and so in Lorentzian signature, where $(x^{AA'})^\dagger = x^{AA'}$, we may be able to cancel terms in our inner product Equation (3.8). This will be done more explicitly in Section 3.3.4.

[7] This notation clearly does not mean the natural numbers but stands for "null". We stick with this notation for consistency with other notation used, as hopefully no confusion should arise.

[8] Again we have used that the ordering of indices is not important to us, i.e. $x^{A'A} \cong x^{AA'}$.

3.2 Projective Spaces

Before discussing the geometrical implications of the incidence relation, we first need the notion of a projective space. A projective space is essentially an equivalence relation on some vector space. For the rest of this project, unless otherwise specified, we shall take the underlying field for this vector space to be the complex numbers.

> **Definition.** [Projective Space] Let V be a complex vector space. Then we define the *projective space* $\mathbb{P}V$ to be the set of equivalence classes on $V \setminus \{0_V\}$ where the equivalence relation is
>
> $$x \sim y \qquad \iff \qquad \exists r \in \mathbb{C}^* \quad : \quad x = ry. \tag{3.11}$$
>
> That is, there is a non-zero complex number that scales y to x.

Claim 3.2.1. If V is a finite dimensional vector space with $\mathbb{C}$-dimension n, then $\dim_{\mathbb{C}}(\mathbb{P}V) = n-1$.

We do not prove the above claim, but the idea is hopefully clear: the equivalence relation Equation (3.11) basically 'compresses' one $\mathbb{C}$ direction. This is why it is called a *projective* space.

Claim 3.2.2. If V is a topological space, then the projective space $\mathbb{P}V$, once equipped with the so-called quotient topology, is also a topological space.[9]

Notation. If V is a product space, i.e. $V = F^{n+1}$, for some field F, then we often denote the corresponding projective space by $F\mathbb{P}^n$, where we note that $(n+1) \to n$ between the two.

We shall denote coordinates on our projective spaces with the standard notation $Z^\alpha := (Z^1, ..., Z^n)$. Once we impose the equivalence relation above, these are actually so-called *homogeneous coordinates*. That is Z^α is a homogeneous coordinate if

$$Z^\alpha \neq (0, ..., 0) \quad \text{and} \quad Z_1^\alpha \sim Z_2^\alpha, \tag{3.12}$$

where the equivalence relation is Equation (3.11).

We will be mainly interested in projective twistor space, $\mathbb{PT}$, which will be related to $\mathbb{CP}^3$:

$$\boxed{\mathbb{CP}^3 := \{Z^\alpha \in \mathbb{C}^4 \,|\, Z^\alpha \neq (0,0,0,0) \quad \& \quad rZ^\alpha \sim Z^\alpha, \forall r \in \mathbb{C}^*\}.} \tag{3.13}$$

We said *related to* above, the reason for this will be explained shortly, but for now we simply say that $\mathbb{PT}$ is actually given by an *open subset* of $\mathbb{CP}^3$. The question of "which open set?" is related to the conformal structure of the spacetime.[10]

3.2.1 Riemann Sphere

There is another, very important, projective space that will appear in what follows, $\mathbb{CP}^1$. This is the equivalence class of complex lines in $\mathbb{C}^2$, with equivalence relation Equation (3.11). This is a 1-complex-dimensional space, and so a 2-real-dimensional space. By considering the charts

$$U_1 := \{(Z^1, Z^2) \in \mathbb{C}^2 \,|\, Z^1 \neq 0\} \quad \text{and} \quad U_2 := \{(Z^1, Z^2) \in \mathbb{C}^2 \,|\, Z^2 \neq 0\},$$

we can show (by considering the *stereographic projection*) that this space is isomorphic to the unit 2-sphere embedded into $\mathbb{R}^3$. For this reason we call the projective space $\mathbb{CP}^1$ the *Riemann sphere*. Riemann spheres are also, of course, just complex lines, and so in this project we will use both "Riemann sphere" and just "line", when the context is clear.

We should be familiar from our experiences with CFT that the group of automorphisms on the unit sphere are exactly the Möbius transformations. This comment is made here as it hints at an important relationship that will follow: Riemann spheres are somehow related to conformal structures.

[9] For more information on quotient topology, see, e.g., Lecture 4 of [5].
[10] This is why we have been writing "(conformally) flat" above.

3.3 Twistor Correspondence

We now want to look at the geometrical links between complex Minkowski spacetime and twistor space. This ultimate goal falls under the name of the *twistor correspondence*. It essentially boils down to the incidence relation. However, as we have introduced the necessary mathematical terminology, we present a formal definition in terms of a double fibration. For now we will drop this and focus simply on using Equation (3.10), however later we will return to this double fibration picture.

The twistor correspondence is encoded in the *double fibration* of the projective spinor bundle, $\mathbb{PS}$,

$$\begin{tikzcd} & \mathbb{PS} \arrow[dl, "\pi_2"'] \arrow[dr, "\pi_1"] & \\ \mathbb{PT} & & \mathbb{M}_{\mathbb{C}} \end{tikzcd} \tag{3.14}$$

where $\mathbb{PS}$ is coordinatised by $(x^{AA'}, \lambda_{B'})$[11] with equivalence relation $\lambda_{B'} \sim r\lambda_{B'}$ for all $r \in \mathbb{C}^*$ [16]. Recalling that the spinor $\lambda_{B'}$ is 2-dimensional complex valued, we see that the equivalence class $\lambda_{B'} \sim r\lambda_{B'}$ acts as a coordinate on the projective space $\mathbb{CP}^1$, i.e. a Riemann sphere. From here it is clear that $\mathbb{PS} \cong \mathbb{M}_{\mathbb{C}} \times \mathbb{CP}^1$, and so we have the simple projection

$$\begin{aligned} \pi_1 : \mathbb{PS} &\to \mathbb{M}_{\mathbb{C}} \\ (x^{AA'}, \lambda_{B'}) &\mapsto x^{AA'}. \end{aligned}$$

Now note that if we define

$$\begin{aligned} \pi_2 : \mathbb{PS} &\to \mathbb{PT} \\ (x^{AA'}, \lambda_{B'}) &\mapsto (ix^{AB'}\lambda_{B'}, \lambda_{A'}) \end{aligned}$$

we get a point in $\mathbb{PT}$ and have also imposed the incidence relation Equation (3.10); points in $\mathbb{PT}$ are given by $(\omega^A, \lambda_{A'})$ with $\omega^A = ix^{AA'}\lambda_{A'}$ (and then subject to the equivalence relation, which $\mathbb{PS}$ has taken care of).

The reason we present this statement of the twistor correspondence is it highlights that it really is a very geometrical object, as fibre bundles are geometrical objects. However, as we said, we shall forget about this formal definition for a while, and simply take the incidence relation itself as the twistor correspondence and see what it can tell us.

3.3.1 $\mathbb{M}_{\mathbb{C}}$ to $\mathbb{PT}$

Ok firstly let's look at what the incidence relation can tell us about how a point in complexified Minkowksi spacetime is mapped to projective twistor space. That is, we want to fix some $x^{AA'} \in \mathbb{M}_{\mathbb{C}}$ and ask what this corresponds to in $\mathbb{PT}$. Well recall that $\mathbb{T}$ itself is (a subset of) $\mathbb{C}^4$, which we coordinatise with $(\omega^A, \pi_{A'})$. The incidence relation, $\omega^A = ix^{AA'}\pi_{A'}$, relates two of these four coordinates to the other two (for fixed $x^{AA'}$), and so we are left with some subset of $\mathbb{C}^2$. We then use our previous comment that the projective space of K^{n+1} is $K\mathbb{P}^n$, to see that the resulting projective space is $\mathbb{CP}^1$. Finally recalling that the incidence relation is linear and holomorphic, we conclude[12]

> A fixed point $x^{AA'} \in \mathbb{M}_{\mathbb{C}}$ corresponds to a linearly and holomorphically embedded Riemann sphere $L_x \cong \mathbb{CP}^1 \subset \mathbb{PT}$ [16].

We have introduced the notation where the corresponding line (i.e. Riemann sphere) to a point $x^{AA'}$ is denoted L_x. The main thing we want to highlight here is that this correspondence is non-local, in the sense

[11] We use $\lambda_{B'}$ for our spinor here to avoid confusion with the projections $\pi_{1/2}$.

[12] It turns out we can actually slightly generalise the above result to say "*any* holomorphic linear embedding of a Riemann sphere in (a subset of) $\mathbb{CP}^3$ can be put into the form of an incidence relation for some fixed $x^{AA'}$" [16]. The proof of this follows from the fact that $\mathbb{CP}^1$ has the Möbius group as its automorphism group along with the projective rescaling constraint, see section 1.4 of [16] for more details.

that a local object (a fixed point in $\mathbb{M}_\mathbb{C}$) is mapped to an extended object (a line in $\mathbb{PT}$). This kind of non-locality is one of the big features in twistor theory.

3.3.2 $\mathbb{PT}$ to $\mathbb{M}_\mathbb{C}$

We now want to go the other way; what does a point in $\mathbb{PT}$ correspond to in $\mathbb{M}_\mathbb{C}$? Well we have already seen that a line in $\mathbb{PT}$ corresponds to a fixed point in $\mathbb{M}_\mathbb{C}$, and we can always represent a point in $\mathbb{PT}$ as the intersection of two (different) lines. This is exactly what we do.

Let $L_x, L_y \in \mathbb{PT}$ be lines that intersect at point $Z^\alpha \in \mathbb{PT}$. From from the incidence relation, we have

$$\omega^A = ix^{AA'}\pi_{A'} \quad \text{and} \quad \omega^A = iy^{AA'}\pi_{A'},$$

where $x, y \in \mathbb{M}_\mathbb{C}$ are the corresponding points to the lines L_x, L_y. Subtracting these two from each other, and using that $x^{AA'} \neq y^{AA'}$ (as otherwise the two lines are identical and we don't have a single point in $\mathbb{PT}$), we obtain

$$(x-y)^{AA'}\pi_{A'} = 0 \qquad \Longrightarrow \qquad (x-y)^{AA'} \propto \pi^{A'},$$

where the implication arrow follows from our inner product being antisymmetric, i.e. $\pi^{A'}\pi_{A'} := \pi_{B'}\epsilon^{B'A'}\pi_{A'}$, but $\epsilon^{B'A'}$ is antisymmetric. In particular we have $(x-y)^{AA'} = \lambda^A\pi^{A'}$, where λ^A is an arbitrary spinor [14]. Note that, in the particular case of Lorentzian $\mathbb{M}$, the reality condition $x^{AA'}, y^{AA'} \in \mathbb{R}$ imposes $\lambda^A = \pi^A$. We will use this later when specialising to such cases.

Now we note that $(x-y)^{AA'}$ represents a vector (i.e. the tangent to the geodesic connecting x and y), and we recall that a null vector is decomposed as the outer-product of two spinors of opposite chirality, Equation (2.7), we see that x and y are *null separated*.

Two lines in $\mathbb{PT}$ intersect if and only if their corresponding $\mathbb{M}_\mathbb{C}$ points are null separated.

Of course we could now use a different line L_w which also intersects L_x and L_y at Z^α, and will again see that w is null separated from both x and y. This essentially corresponds to varying over λ^A, and the result is a 2-dimensional totally null plane in $\mathbb{M}_\mathbb{C}$, i.e. every tangent vector to this plane is null and *totally* meaning no spacelike vectors. It also turns out that the tangent *bivectors* to this plane are self-dual. We call such a plane an *α-plane* [14]. If we then repeat the whole procedure, but using the dual incidence relation, we obtain β-planes, i.e. 2-dimensional null planes who's bivectors are *anti*-self-dual.[13]

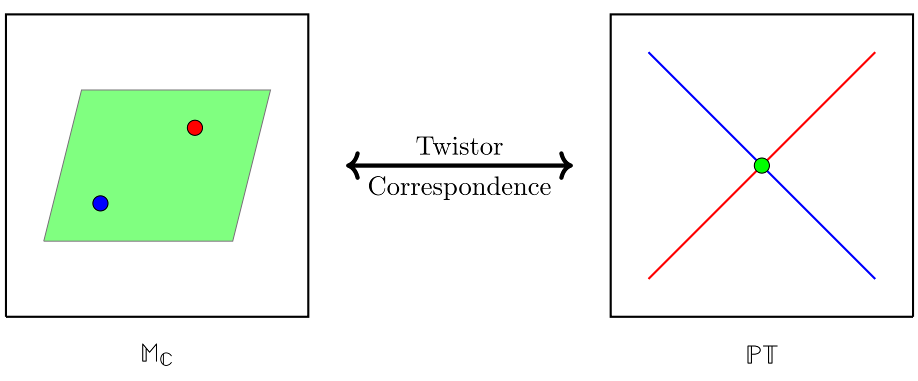


Figure 3.1: *Pictorial description of the twistor correspondence. Points in complexifed Minkowski spacetime, $\mathbb{M}_\mathbb{C}$, are mapped to embedded Riemann spheres, or lines, in projective twistor space $\mathbb{PT}$. Conversely, a point in $\mathbb{PT}$ is mapped to a totally null 2-plane in $\mathbb{M}_\mathbb{C}$, who's tangent bivectors are self-dual. Such a plane is known as an α-plane.*

[13] We should remark that while α-planes correspond to points in $\mathbb{PT}$, β-planes correspond to 2-planes in $\mathbb{PT}$. This will not be too important for us, though.

3.3.3 Null Cones & Conformal Structure

A nice discussion of α/β-planes can be made by considering something known as the *Klein quadric*, Q_4, and the *Plücker embedding* of $\mathbb{M}_{\mathbb{C}}$ into $\mathbb{CP}^5$. Unfortunately the discussion of this, naturally involving a lot of mathematical formalism, will take us too far away from our current goals, and so the interested reader is instead referred to [4]. The important thing we want from this discussion is that any null line in $\mathbb{M}_{\mathbb{C}}$ is the intersection of an α-plane and a β-plane (see Theorem 1.4.1 of [4]). Putting this together with the fact that α/β-planes are totally null in $\mathbb{M}_{\mathbb{C}}$, we conclude that the null cones in $\mathbb{M}_{\mathbb{C}}$ are fibred by α- and β-planes.

We now note that two points $x, y \in \mathbb{M}_{\mathbb{C}}$ are null separated if only if y lies on x's null-cone[14] (or vice versa). So the above correspondence tells us that the structure of the null-cone at $x \in \mathbb{M}_{\mathbb{C}}$ is identified in $\mathbb{PT}$ by lines that intersect L_x. Putting this together with the fact that the allocation of a null-cone in every tangent space of a manifold gives us a conformal structure, we see that the conformal structure of $\mathbb{M}_{\mathbb{C}}$ is encoded in our twistor correspondence.

Finally recalling that the correspondence has been purely holomorphic (i.e. no barred spinors have appeared), we arrive at another 'moral' of twistor theory: holomorphic structures on $\mathbb{PT}$ encode conformal structures on $\mathbb{M}_{\mathbb{C}}$ [16]. As we will shortly explain, we are being a bit too keen here, as null-cones only determine the conformal structure *up to boundary conditions*. So really, all we have right now is that the *conformal class* is captured by holomorphic structures on $\mathbb{PT}$.

3.3.4 Space Of Null Twistors & Lorentzian Minkowski Spacetime

We can now show an interesting result, which we shall use shortly. Firstly we note that if we are given a solution to the incidence relation, i.e. given a $x_0^{AA'}$ such that

$$\omega^A = i x_0^{AA'} \pi_{A'},$$

we get another solution simply by

$$x^{AA'} = x_0^{AA'} + \lambda^A \pi^{A'}, \tag{3.15}$$

where the spinor λ^A is the same as the one above — i.e. if we denote $x_0^{AA'}$ by $y^{AA'}$ and we get the above result.

We get our α-plane by varying λ^A in Equation (3.15). In general, of course, an α-plane contains no real points, as we are working with $\mathbb{M}_{\mathbb{C}}$. However we now ask the question "what if it does?" W.l.o.g. we take $x_0^{AA'} \in \mathbb{R}$ to be such a real point on our α-plane. Now contract Equation (3.15) with $\pi_{A'}$,

$$x^{AA'}\pi_{A'} = x_0^{AA'}\pi_{A'},$$

but the left-hand side is (up to a factor of i) just ω^A, by the incidence relation. If we make this substitution and further contract with $\bar{\pi}_A$, we get

$$\omega^A \bar{\pi}_A = i x_0^{AA'} \bar{\pi}_A \pi_{A'}.$$

Now comes the interesting point: taking the conjugate of this we get

$$\bar{\omega}^{A'} \pi_{A'} = -i x_0^{AA'} \bar{\pi}_A \pi_{A'},$$

where we have used that $x_0^{AA'} \in \mathbb{R}$. However if we then substitute these into our inner product Equation (3.8), we conclude that *the corresponding twistor is null*!

So we have that if the α-plane contains a real point the corresponding twistor is null. What about the reverse? That is, what does a twistor being null tell us about the α-plane? Well it follows from the calculation that we just did that a null twistor must obey

$$\omega^A \bar{\pi}_A = ia$$

for some $a \in \mathbb{R}$. If we then define

$$x_0^{AA'} = \frac{1}{a}\omega^A \bar{\omega}^{A'},$$

[14]For clarity, this is the complex null cone of $\mathbb{M}_{\mathbb{C}}$, not simply the light-cones of Lorentzian Minkowski.

which is real, we then see immediately that

$$\omega^A = i x_0^{AA'} \pi_{A'}.$$

So in total we conclude[14]: an α-plane contains a real point if and only if the corresponding twistor is null, and the α-plane contains the whole null geodesic

$$x^{AA'} = x_0^{AA'} + r\bar{\pi}^A \pi^{A'}, \qquad r \in \mathbb{R}.$$

So why is this particularly interesting to us? Well recall that earlier we complexified Minkowski spacetime using the motto that we would "ask questions later". We said all the way back in Section 2.2.2 that we can get our different signatures by imposing certain reality conditions. In slightly more technical language, we take a real hypersurface of $\mathbb{M}_{\mathbb{C}}$, and which real hypersurface we take determines the resulting signature.

This is exactly what we have just done; we have the hypersurface corresponding to α-planes containing a real point. We have just shown that these correspond to the null twistors, which once we impose our projective rescaling gives us the space

$$\mathbb{PN} := \{Z \in \mathbb{PT} \,|\, \Sigma(Z) = 0\} \tag{3.16}$$

known as "the space of null twistors".

Now recall that, for Lorentzian signature, our dual twistors are obtained by Equation (3.9), with the further constraint $x^{AA'} \mapsto (x^{AA'})^\dagger = x^{AA'}$. Putting this latter constraint together with the results above, we can conclude that

The space of null twistors, $\mathbb{PN}$, corresponds to the real, Lorentzian signature, hypersurface of $\mathbb{M}_{\mathbb{C}}$.

We see that the above construction tells us that the intersection of an α-plane with the Lorentzian hypersurface $\mathbb{M} \subset \mathbb{M}_{\mathbb{C}}$ gives us a single null geodesics. Note it is *not* a 2-plane of null geodesics, as the 2-planeness came from our freedom in λ^A, however the condition $\lambda^A = (\pi^{A'})^\dagger =: \bar{\pi}^A$ completely restricts this freedom.

We have only discussed the Lorentzian real hypersurface, but one can make similar arguments to obtain the Euclidean and split signature hypersurfaces. Details of these results can be found in Chapter 2 of [16].

3.4 Conformal Structures

We now return to the comment we made at the end of Section 3.3.2; the fact that holomorphic structures on $\mathbb{PT}$ encode information about the conformal *class* of the spacetime. As we have been mainly interested in Minkowski spacetime, we focus on the class of conformally flat spacetimes.

3.4.1 Compactified Minkowski Spacetime

Recall that a general conformal Killing vector in $d > 2$ dimensions is given by

$$X_a = P_a + M_{ab}x^b + Dx_a - B_a x^2 + 2B_b x^b x_a,$$

where P_a are the spacetime translations, M_{ab} Lorentz transformations, D are dilations, and B_a are the special conformal transformations. If we consider a pure special conformal transformation, then the integral curves of such a conformal Killing vector are simply

$$\frac{dx^a}{ds} = 2B_b x^b x_a - B^a x^2,$$

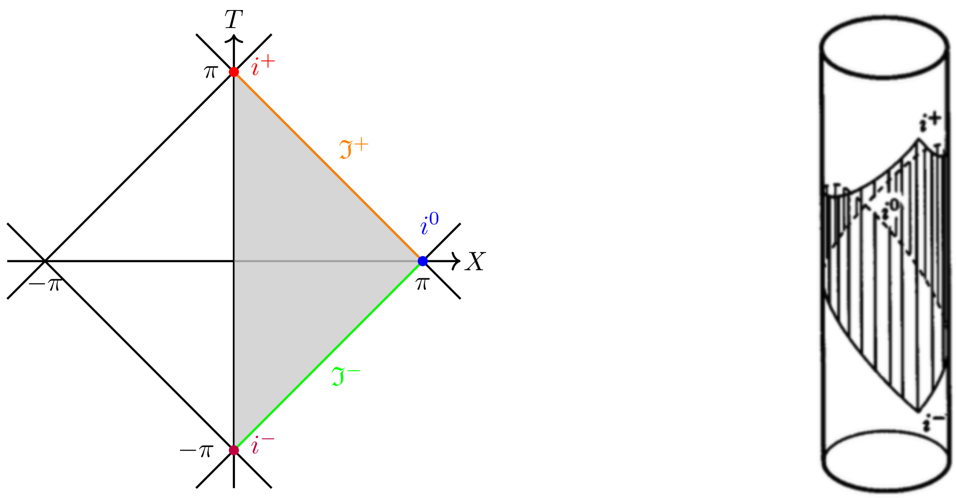
which we can integrate to give [14]

$$x^a(s) = \frac{x^a(0) - sB^a x^2(0)}{1 - 2sB_b x^b(0) + s^2 B^2 x^2(0)}.$$

The important thing to note is that the denominator vanishes for *finite* parameter value, s^*. This tells us that our vector field X^a is *incomplete* on $\mathbb{M}$, as it blows up at finite s^*.

Luckily this is easily fixed, we must simply add in the points at infinity. In doing this, the special conformal transformations then just exchange the points at infinity with other finite points in $\mathbb{M}$. We call this extended spacetime *compactified Minkowski* spacetime, and denote it by $\mathbb{M}^c$. We can examine the topology of $\mathbb{M}^c$ using the standard conformal compactification techniques used in obtaining Penrose diagrams. Details of such techniques can be found in Lecture 23 of [7]. We do not go through the steps here, but simply show the result diagrammatically.

The Penrose diagram for Minkowski spacetime is given by the shaded region in the following diagram:



(a) *Drawing taken from my notes on [7].* **(b)** *Figure from [14].*

Figure 3.2: *Diagrammatic representation of the conformal structure of Minkowski spacetime. The Einstein static universe is topologically $\mathbb{R} \times S^3$, whereas in a) the S^3 is compressed to $\mathbb{R}$ (so the diagram sits on $\mathbb{R}^2$), whereas in b) the ESU is compressed to $\mathbb{R} \times S^1$, the cylinder. The shaded region is then Minkowski spacetime. The labelling of the points is explained below.*

Where we have labelled:

- Spacelike infinity, i^0,
- Future timelike infinity, i^+,
- Past timelike infinity, i^-,
- Future null (or lightlike) infinity $\mathfrak{I}^+$, and
- Past null (or lightlike) infinity $\mathfrak{I}^-$.

These figures let us see that the compactified Minkowski spacetime is a manifold with boundary, the boundary being $\mathfrak{I}^\pm$. If we want to make our space $\mathbb{M}^c$, we simply identify opposite generators of $\mathfrak{I}^-$ and $\mathfrak{I}^+$ [14]. In other words, we identify $i^{\pm,0}$ as a single point, denoted $\mathfrak{I}$. We call $\mathfrak{I}$ the *conformal infinity* of the spacetime, and it is exactly this that allows us to differentiate different spacetimes in the same conformal class: for Minkowski spacetime we have just shown that $\mathfrak{I}$ consists of 3 points; whereas for 4-dimensional de Sitter space — which is conformally flat — $\mathfrak{I}$ is given by two spacelike S^3s [16].

3.4.2 Twistor Space Is Blind To Conformal Structure

We now want to show explicitly that twistor space is blind to the conformal structure; that is the holomorphic structures on twistor space only tell us about the conformal *class* of our spacetime. Essentially this amounts to us trying to find a linear action of the complexified 4-dimensional conformal group, $SL(4,\mathbb{C})$. Such an action is given by the generators T^α_β, acting as $Z^\alpha \mapsto T^\alpha_\beta T^\beta$. The claim is that the linear generators

$$T^\alpha_\beta = Z^\alpha \frac{\partial}{\partial Z^\beta}$$

do the job. We get the standard conformal generators via [16]

$$P_{AA'} = \omega_A \frac{\partial}{\partial \pi^{A'}}, \qquad J_{AB} = \omega_{(A} \frac{\partial}{\partial \omega^{B)}}, \qquad \widetilde{J}_{A'B'} = \pi_{(A'} \frac{\partial}{\partial \pi^{B')}},$$
$$K^{AA'} = \pi^{A'} \frac{\partial}{\partial \omega_A}, \quad \text{and} \quad D = \frac{1}{2}\left(\omega_A \frac{\partial}{\partial \omega_A} - \pi^{A'} \frac{\partial}{\partial \pi^{A'}}\right),$$

which we can verify satisfy the conformal algebra.

So we have constructed an explicit linear action of the conformal group on $\mathbb{PT}$, which means projective twistor space is conformally invariant. This is just the statement that $\mathbb{PT}$ cannot distinguish between two spacetimes in the same conformal class. From here it is clear that, if we want to be able to single out $\mathbb{M}_\mathbb{C}$ itself, we need some structure on $\mathbb{PT}$ which is *not* conformally invariant. Putting this together with the discussion of the last subsection, it's clear that this additional structure must have something to do with the points at infinity in $\mathbb{PT}$. This is exactly what we construct now.

As $\mathbb{M}_\mathbb{C}$ is our ultimate goal, we focus on spacetimes that are conformally flat. The general line element is given by[15]

$$ds^2 = \frac{1}{\big(f(x)\big)^2} dx^{AA'} dx_{AA'}. \tag{3.17}$$

Clearly we get $\mathbb{M}_\mathbb{C}$ when $f(x) = 1$. This is written in spacetime language, but we want to study twistor space, so the first thing we do is translate this into twistor variables.

The first thing we note is that any line can be defined by specifying *any* two points on it. In terms of $\mathbb{PT}$ this translates to us being able to specify any line L_x by the antisymmetric product of two points Z_1^α, Z_2^β that lie on the line, i.e. [16]

$$X^{\alpha\beta} = Z_1^{[\alpha} Z_2^{\beta]}.$$

Then using Equation (3.7) and the incidence relations, we can rewrite this as[16]

$$X^{\alpha\beta} = \omega_1^C \omega_{2,C} \begin{pmatrix} \epsilon_{AB} & -x_A^{B'} \\ x_B^{A'} & \frac{1}{2}\epsilon^{A'B'} x^2 \end{pmatrix}. \tag{3.18}$$

So we see that $X^{\alpha\beta}$ encodes a point in spacetime up to a scale given by $\omega_1^A \omega_{2,A}$.

We then start from the natural choice of line element

$$ds_X^2 = \epsilon_{\alpha\beta\gamma\delta} dX^{\alpha\beta} dX^{\gamma\delta}, \tag{3.19}$$

and the aim is to see how it relates to Equation (3.17). Plugging in Equation (3.18) and going through the algebra we quickly arrive at

$$ds_X^2 = \big(\omega_1^A \omega_{2,A}\big)^2 dx^{AA'} dx_{AA'},$$

and so we see straight away that ds_X^2 is conformally flat with conformal factor

$$f(x) = \big(\omega_1^A \omega_{2,A}\big)^{-1}.$$

However we have been a bit sloppy: we said above that $X^{\alpha\beta}$ only encodes a point of spacetime *up to scale*, and so we must consider it projectively if we want to use them as coordinates for spacetime. This is the statement that we must treat them as *homogeneous coordinates*. The problem is then that ds_X^2 has homogeneous weight $+2$ and so is not projectively well-defined, i.e. there is nothing to cancel the scaling of

[15] We use $f(x)$ to denote the conformal factor rather than $\Omega(x)$, in order to avoid confusion with the twistor $\Omega(x)$.

[16] The matrix here is flipped compared to (1.29) in [16] as we our definition of Z^α has the unprimmed spinor first, whereas they have the primed spinor first.

$X^{\alpha\beta}$ and $X^{\gamma\delta}$, which we require as the line element is meant to give a physical length. We therefore must modify ds_X^2 to give us a projectively well-defined line element. The obvious choice is

$$ds_I^2 = \frac{\epsilon_{\alpha\beta\gamma\delta}\, dX^{\alpha\beta} dX^{\gamma\delta}}{(I_{\alpha\beta}X^{\alpha\beta})^2}, \tag{3.20}$$

where $I_{\alpha\beta}$ is some fixed antisymmetric bi-twistor, i.e. it is a twistor 2-form. $I_{\alpha\beta}$ is known as the *infinity twistor* and it is exactly the structure we need to specify in order to break our conformal invariance and so distinguish two spacetimes in the same conformal class. Let's see why this is the case.

Equation (3.20) is clearly singular when $I_{\alpha\beta}X^{\alpha\beta} = 0$. This hypersurface defines the points at infinity in the usual conformal compactification sense [16]. We can demonstrate this explicitly for the Minkowski case. Consider the infinity twistor

$$I_{\alpha\beta} = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & \epsilon^{AA'} \end{pmatrix}.$$

Contracting with Equation (3.18), we simply get

$$I_{\alpha\beta}X^{\alpha\beta} = \frac{1}{2}\omega_1^B \omega_{2,B}\epsilon^{AA'}\epsilon_{AA'} = \omega_1^A \omega_{2,A} \tag{3.21}$$

where we have used $\epsilon^{AA'}\epsilon_{AA'} = 2$ and relabelled $B \to A$. Putting this together with Equation (3.19) we see these factors cancel and we are simply left with

$$ds^2 = dx^{AA'}dx_{AA'},$$

which confirms that we are dealing with the infinity twistor for Minkowski spacetime.

We now need to show that $I_{\alpha\beta}$ captures all the structure at infinity. It follows from Equation (3.21), and the antisymmetry of our spinor inner product, that our hypersurface $I_{\alpha\beta}X^{\alpha\beta} = 0$ corresponds to $\omega_1^A \propto \omega_2^A$, or that at least one of them vanishes. Let's assume neither vanish, then from the incidence relations we have $\pi_{1,A'} \propto \pi_{2,A'}$, where the proportionality is the same as for the ωs. So in total we have

$$Z_1^\alpha \propto Z_2^\alpha,$$

but these Zs lie on the line L_x and so should be understood projectively, and so must correspond to *the same point* on L_x. This is now a contradiction because, in order to define L_x in the first place, we needed $Z_1^\alpha \neq Z_2^\alpha$.

So we see that at least one ω_i^A must vanish. W.l.o.g. let's assume it is ω_1^A so that $Z_1^\alpha = (0, \pi_{1,A'})$. It then follows immediately from the incidence relations that, if $x^{AA'}$ is everywhere finite, $\pi_{1,A'} = 0$. This would then give us $Z_1^\alpha = (0,0)$, but this is excluded in our definition of a homogeneous coordinate, Equation (3.12). So the only way we can resolve this problem is to allow $x^{AA'}$ to be infinite, which is exactly what we wanted! That is, lines in $\mathbb{PT}$ which satisfy $I_{AB}X^{AB} = 0$ correspond exactly to the points at infinity in $\mathbb{M}_{\mathbb{C}}$.

Indeed it is also true that the infinity twistor for Minkowski spacetime is the line in $\mathbb{PT}$, denoted I, corresponding to spacelike infinity of $\mathbb{M}_{\mathbb{C}}$, i^0 [16]. Putting this together with the fact that we know that the intersection of two lines in $\mathbb{PT}$ correspond to null separated points in $\mathbb{M}_{\mathbb{C}}$, we see that any line that intersects I corresponds to a point null separated from i^0, but these are just the boundaries $\mathfrak{I}^\pm$. So in total we see that the infinity twistor encodes *all* the information about the points of infinity in $\mathbb{M}_{\mathbb{C}}$, and so encodes the conformal structure.

Finally recall that we said just before Section 3.2.1 that $\mathbb{PT}$ is given by an open subset of $\mathbb{CP}^3$. Well we can now answer the question of "which open subset?" Well recall that we want $\mathbb{PT}$ to correspond to the *un*compactified Minkowski spacetime, which does *not* include the points at infinity. Putting this together with the fact that we've just shown that points at infinity of spacetime are given in twistor space by $I_{\alpha\beta}X^{\alpha\beta} = 0$, we conclude that $\mathbb{PT}$ is given by the open subset of $\mathbb{CP}^3$ satisfying $I_{\alpha\beta}X^{\alpha\beta} \neq 0$ [16].

3.5 Kerr Theorem

Now that we have a understanding of twistor space and it's geometrical links with spacetime, we take a step back and talk about null geodesic congruences in order to obtain the important notion of *shear*.

3.5.1 Geodesic Congruences

The idea of a null congruence should be familiar from GR. We define a congruence as the integral curves of some nowhere vanishing vector field. If these integral curves are then geodesics, we obtain a geodesic congruence, and if these geodesics are in turn null, we get our null geodesic congruence. Intuitively, they are simply a set of null geodesics each one passing through some region $U \in \mathcal{M}$.[17] We denote the null congruence as Γ and the individual geodesics by γ_τ, where τ is a smooth one-parameter label describing which geodesic we are talking about. As we just explained, we can describe Γ by the null vector fields which the γ_τs are integral curves of. We denote these vector fields ℓ^a_τ, where a is the tensor index. As the geodesics are null, we will get the same congruence if we scale them, and so we are free to scale our ℓ^a_τ.

Geodesic congruences find massive use in GR as they allow us to ask about *geodesic deviation.* The idea is to define a so-called *Jacobi field* via[18]

$$\eta(x) := \frac{d\gamma_\tau}{d\tau}\bigg|_{\tau=0},$$

which tells us how the geodesics neighbouring $\gamma := \gamma_0$ behave.[19] In other words, $\eta(x)$ can tell us whether the geodesics converge or not. This has a clear intuitive link to the curvature of $\mathcal{M}$, and indeed one can show — see e.g. Section 3.3.4 of [17] — that

$$D^2\eta^a \equiv \ell^b\nabla_b\big(\ell^c\nabla_c\eta^a\big) = R^a{}_{bcd}\ell^b\eta^c\ell^d \tag{3.22}$$

holds, where $R^a{}_{bcd}$ is the Riemann tensor and we have defined $D := \ell^a\nabla_a$. Jacobi fields also satisfy

$$D\eta^a = \eta^b\nabla_b\ell^a. \tag{3.23}$$

3.5.2 Complex Shear

Now comes the interesting bit, we have demonstrated that we can trade a real null vector field with a spinor field via Equation (2.7). We do just this, and denote the spinor corresponding to ℓ^a by o^A. The geodesic equation translates to[20] [14]

$$D\ell^a = k\ell^a \qquad \Longrightarrow \qquad o^A Do_A = 0. \tag{3.24}$$

We then use that we can fix the scale of ℓ^a, and so the scale of o^A, to demand that each geodesic actually obeys

$$Do_A = 0. \tag{3.25}$$

Recalling that o_A defines a null flag, i.e. a flagpole and flagplane, we see that Equation (3.25) corresponds to parallel transport of the null flag along γ.

We now introduce another spinor, ι^A satisfying

$$o_A\iota^A = 1 \quad \text{and} \quad D\iota^A = 0. \tag{3.26}$$

These two spinors define a dyad in spin space. We then recall Equation (2.14) and define $m^a := o^A\bar{\iota}^{A'}$ and $\bar{m}^a := \iota^a\bar{o}^{A'}$, which we have already shown are complex null vectors. We can then use Leibniz along with Equations (3.25) and (3.26) to obtain

$$Dm^a = 0 \quad \text{and} \quad D\bar{m}^a = 0. \tag{3.27}$$

So we see that these complex null vectors are parallelly propagated along our null curve γ. On top of this, they are both orthogonal to ℓ^a. Now, as m^a and $\bar{m}^a$ are linearly independent, they span a null 2-plane which is orthogonal to γ. We call this space *screen-space* and denote it S.

[17] We work with a generic spacetime manifold $\mathcal{M}$ until otherwise specified.

[18] This $\eta(x)$ is not the Minkowski metric. This notation is just used to make comparisons with [14] easier. Other standard notation exists, including $J(x)$ and $S(x)$.

[19] We will now focus on this specific geodesic, and so drop the τ label.

[20] Note that here we have used that we can uniquely extend our spacetime connection to one on spinors, as described at the start of Section 3.1, and so the D acting on o_A is well defined.

Next we note that if η^a is orthogonal to ℓ^a, our neighbouring geodesics are "abreast in time". That is, the hypersurface that η^a lies in defines a time-slice (from γ's rest frame) of the congruence. This hypersurface is exactly our screen space, and so such a Jacobi field can always be written as

$$\eta^a = \bar{z}m^a + z\bar{m}^a \qquad z \in \mathbb{C}.$$

We now use Equations (3.23) and (3.27), and contract with m_a (using $m_a m^a = 0$, as it's null) to obtain

$$\bar{m}^a m_a Dz = zm^a\bar{m}^b\nabla_b\ell_a + \bar{z}m^a m^b \nabla_b \ell_a.$$

Now we recall that $\bar{m}^a m_a = -1$, Equation (2.15), to finally obtain

$$Dz = -\rho z - \sigma\bar{z}, \tag{3.28}$$

where we have defined [14]

$$\begin{aligned} \rho &= m^a\bar{m}^b\nabla_b\ell_a = o^A\bar{m}^b\nabla_b o_A, \\ \sigma &= m^a m^b\nabla_b\ell_a = o^A m^b\nabla_b o_A. \end{aligned} \tag{3.29}$$

The important thing we have just shown is that the behaviour of neighbouring geodesics relative to each other is given by two complex functions, ρ and σ. We now want to investigate the effects of each.

Consider a circle, C, of geodesics around γ initially given by

$$z = \epsilon e^{i\varphi}$$

for some small parameter ϵ and $0 \leq \varphi < 2\pi$. Moving a small parameter distance $\delta\lambda$ along γ gives

$$\delta z = -(\rho z + \sigma\bar{z})\delta\lambda. \tag{3.30}$$

We now consider the effects of ρ and σ separately:

(i) First consider the case when $\sigma = 0$ and $\rho = Re^{i\psi}$, where $R \in \mathbb{R}^+$. Then Equation (3.30) gives

$$z = \epsilon e^{i\varphi} \mapsto z + \delta_\rho z = \epsilon e^{i\varphi}(1 - Re^{i\psi}\delta\lambda).$$

Physically, this corresponds to the radius of C decreasing by a factor $(1 - R\cos(\psi)\delta\lambda)$ and rotating (relative to γ) by a factor $R\sin(\psi)\delta\lambda$.

So we see that ρ captures information about the convergence and rotation of C.

(ii) Now let's consider the case $\sigma = |\sigma|e^{2i\psi}$ and $\rho = 0$. Then Equation (3.30) gives

$$z \mapsto z + \delta_\sigma z = \epsilon\big(e^{i\varphi} - |\sigma|e^{i(2\psi-\varphi)}\delta\lambda\big). \tag{3.31}$$

It is less easy to see what this means, and so we need to manipulate it further. The idea is that $z \in \mathbb{C}$ and so we can express the above as

$$z + \delta_\sigma z = e^{i\psi}(x + iy),$$

where the factor of $e^{i\psi}$ is included to make the result nicer (it is simply a rotation of the (x, y) coordinates and so changes nothing). If we substitute Equation (3.31) into this and then solve for x and y we obtain

$$x = \epsilon(1 - |\sigma|\delta\lambda)\cos(\varphi - \psi) \quad \text{and} \quad y = \epsilon(1 + |\sigma|\delta\lambda)\sin(\varphi - \psi).$$

This is the equation for an ellipse with major/minor axes $\epsilon(1 \pm |\sigma|\delta\lambda)$, respectively. On top of this we note that, to this order, the area of C is unchanged. So we have essentially "squashed" C into an ellipse. This is clearly just a kind of shear, and we call σ the *complex shear*.

$\sigma = 0,\ \rho = Re^{i\psi}$ $\qquad$ $\sigma = |\sigma| e^{2i\psi},\ \rho = 0$

Figure 3.3: *Effects of ρ and σ (Equation (3.29)) on a circle of null geodesics, C, in a geodesic congruence, as we move a small parameter distance $\delta\lambda$ along the centred geodesic. Left: ρ controls the convergence and rotation of C, as indicated by the arrows. The convergence is given by $(1 - R\cos(\psi)\delta\lambda)$ and the rotation by $R\sin(\psi)\delta\lambda$. Right: σ is the complex shear and results in an ellipse of equal area to C. The major/minor axes are $\epsilon(1 \pm |\sigma|\delta\lambda)$, respectively.*

The interesting quantity to us is the complex shear. Using the results from above, and introducing the spinor versions of the Weyl tensor and Ricci curvature, we can show (see chapter 6 of [14]) that the null hypersurface generated by our null vectors ℓ^a admit a conformal metric if and only if the generators of the hypersurface are shear-free. Such generators are called *geodesic-shear-free* (g.s.f.), and must obey

$$o^A D o_A = o^A o^B \bar{o}^{B'} \nabla_{BB'} o_A = 0, \quad \text{and} \quad \sigma = o^A o^B \bar{\iota}^{B'} \nabla_{BB'} o_A = 0,$$

which are just the geodesic, Equation (3.24), and vanishing shear, Equation (3.29), conditions expressed purely in terms of spinors (i.e. we used $m^a = o^A \bar{\iota}^{A'}$ etc). We can clearly just combine these two conditions into the single g.s.f. condition

$$o^A o^B \nabla_{BB'} o_B = 0. \tag{3.32}$$

The other important thing to note about shear is that is has a link to the complex analyticity of our congruences. We can see this from the above condition by showing that there is a link between a 2-dimensional complex manifold having a conformal metric and having what is known as a *complex structure*. However we can also see the link simply by the fact that if $\sigma = 0$ then the $\bar{z}$ dependence drops out of Equation (3.28); the motion of the congruence on S is holomorphic if and only only if the generators are shear-free.

Before moving on to discuss the Kerr theorem, we should state that it turns out that satisfying the shear-free condition is highly restrictive in non-flat spacetimes [14]. We do not go through the details here, but simply make this point to further justify why we have been considering (conformally) flat spacetimes, and so now again specialise to $\mathcal{M} = \mathbb{M}_{\mathbb{C}}$.

3.5.3 Kerr Theorem

We are now in a place to introduce a very important theorem when it comes to trying to solve the z.r.m. equations, Equation (3.2). It goes by the name *Kerr Theorem*, and we introduce it as follows.

A general analytic function $f(Z^\alpha)$ is *not* well-defined on $\mathbb{PT}$. The reason for this is simply that $\mathbb{PT}$ is defined projectively, and so if we want a well-defined function $f(Z^\alpha)$, we require it to have a definite homogeneous degree, i.e.

$$f(rZ^\alpha) = r^d f(Z^\alpha),$$

where d is the homogeneous degree of f. Suppose we have such a well-defined function, and consider its zero set, i.e. the set $\{Z^\alpha | f(Z^\alpha) = 0\}$. This defines some 3-dimensional (as there are 4 components in Z^α, but we loose one by the zero set condition) hypersurface in $\mathbb{PT}$. If we want to talk about what this set encodes in Lorentzian Minkowski spacetime, we must take the intersection of this hypersurface and $\mathbb{PN}$, as per Section 3.3.4. We denote this intersection by K, and it defines a congruence of null geodesics in $\mathbb{M}$.

The Kerr theorem then states that this congruence, K, is shear-free and that all analytic g.s.f. congruences arise in this way [14].

We do not prove the Kerr theorem here, but a proof (using a tetrad formalism) can be found via [18].

3.5.4 Robinson Congruence

Before moving on to discuss the solutions to the z.r.m. solutions, it is instructive for us to give a simple, but very important, example of the application of Kerr's theorem.

Consider the twistor function

$$f(Z^\alpha) = A_\alpha Z^\alpha,$$

where $A_\alpha = (A_A, A^{A'})$ is a dual twistor. Then our zero set condition gives us

$$\big(iA_A x^{AA'} + A^{A'}\big)\pi_{A'} = 0 \qquad \Longrightarrow \qquad \pi^{A'} = k\big(iA_A x^{AA'} + A^{A'}\big).$$

If you then plug through a bit of slightly messy algebra — see chapter 7 of [14] — we can show the integral curves are circles on a torus. However they are *not* the 'normal' circles defining $T = S^1 \times S^1$ (i.e. the circles that you get by intersecting a torus with the x and y axes), but instead these circles twist around the torus and all link together.

The twisting behaviour is determined by a quantity

$$a := -\frac{1}{2} A_\alpha \bar{A}^\alpha.$$

The details are not important for us, apart from that we note that $a = 0$ iff the dual twistor is null.

- When the dual twistor is null, the congruence degenerates, and so by our "two points are null separated iff the corresponding twistors meet" we see that the spacetime picture corresponds to a congruence of null geodesics which all meet the null geodesic defined by the dual twistor A_α.
- If $a \neq 0$, and so the dual twistor is non-null, the geodesics fail to meet, and instead they twist around the torus, see Figure 3.4. The light rays corresponding to this non-null case are called a Robinson congruence, and it is this twisting nature that gives twistor theory its name.

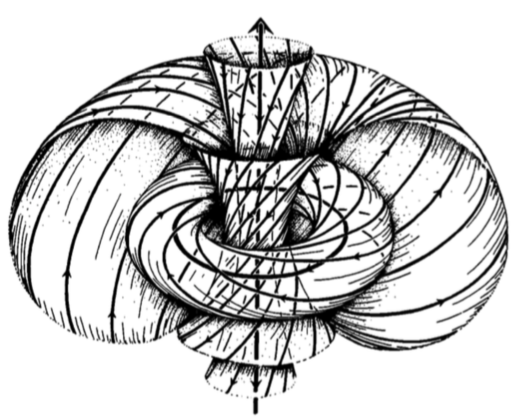


Figure 3.4: *A depiction of a Robinson congruence. A light ray is depicted by a point in* $\mathbb{R}^3$ *with the arrow indicating the direction of motion [11]. Figure taken from [15].*

4 | Sheaf Cohomology

Recall that, all the way back in Section 1.2.4 we introduced the notion of deRham cohomology as the equivalence classes of closed forms modulo exact forms. As we mentioned then, and as the title of this chapter suggests, deRham cohomology is just one particular type of cohomology. While deRham cohomology plays a *huge* role in algebraic topology, twistor theory is more concerned with what is known as *sheaf cohomology*. Just as deRham cohomology measures the number of 'holes' in a space, sheaf cohomology measures the distinction between *local* and *global* information [14]. However in order to properly understand the latter, we need to introduce the terminology of general cohomology theory. We will use the nice geometric notions of deRham cohomology to help keep us grounded. We will work off the definitions/explanations given in [6], as well as the other references provided so far.

4.1 Cohomology Terminology

4.1.1 Exact Sequences

The first notion we need is that of an *exact sequence*. Consider a sequence of vector spaces, $\{V_i\}$, connected by linear maps $\varphi_i : V_i \to V_{i+1}$:

$$\dots \xrightarrow{\varphi_{i-2}} V_{i-1} \xrightarrow{\varphi_{i-1}} V_i \xrightarrow{\varphi_i} V_{i+1} \xrightarrow{\varphi_{i+1}} \dots.$$

We say that the maps are *exact at* V_i if $\text{im}\varphi_{i-1} = \ker \varphi_i$, where im/ ker stand for image and kernel, respectively. It immediately follows from this that if our maps are exact at V_i then $\varphi_i \circ \varphi_{i-1} v = 0$ for all $v \in V_{i-1}$. The sequence itself is called *exact* if it is exact at all V_i.

There is a particularly interesting type of exact sequence, known as a *short exact sequence* given by

$$0 \xrightarrow{\varphi_0} V_1 \xrightarrow{\varphi_1} V_2 \xrightarrow{\varphi_2} V_3 \xrightarrow{\varphi_3} 0, \tag{4.1}$$

where 0 denotes the zero-dimensional vector space containing simply the zero vector. Short exact sequences are interesting as one can easily show[1] that Equation (4.1) is exact iff φ_1 is injective and φ_2 is surjective. In particular this tells us that if the sequence

$$0 \longrightarrow V \xrightarrow{\varphi} W \longrightarrow 0,$$

is exact then $V \cong W$. A sequence that is not short is called *long*.

4.1.2 Differential Complexes

It is not a coincidence that the word "exact" has appeared both in our definition of deRham cohomology and in the context of exact sequences. To make the connection we introduce the notion of a *differential complex*.

Let $C = \oplus_{i\in\mathbb{Z}} C_i$ be a direct sum of vector spaces C_i. If we equip C with a set of linear maps $d_i : C_i \to C_{i+1}$ such that $d_{i+1} \circ d_i = 0$, then we call the complete construction a *differential complex*. We refer to d_i as the *i-th differential of the complex*.[2] Of course we can represent a differential complex as a sequence

$$\dots \xrightarrow{d_{i-2}} C_{i-1} \xrightarrow{d_{i-1}} C_i \xrightarrow{d_i} C_{i+1} \xrightarrow{d_{i+1}} \dots, \tag{4.2}$$

[1] Using the result that injectivity of a map $\phi : V \to W$ is given provided $\ker \phi = \{0_V\}$.

[2] We should note that d_i need not be a derivative in the familiar sense of undergraduate calculus.

subject to $\text{im} d_i \subseteq \ker d_{i+1}$. It is important to note that Equation (4.2) need not be exact, and indeed this is the whole point.

We then define the subspaces

$$Z^i(C) := \ker d_i \quad \text{and} \quad B^i(C) := \text{im} d_{i-1},$$

who's elements are known as *i-cocycles* and *i-coboundaries*, respectively.[3] This notation should look familiar from Section 1.2.4, and we define the *cohomology of C* via

$$H(C) := \bigoplus_{i\in\mathbb{Z}} H^i(C) \qquad \text{where} \qquad H^i(C) := Z^i(C)/B^i(C).$$

The cohomology measures exactly how much the complex Equation (4.2) fails to be exact. We can ground this information by noting that the deRham cohomology is simply the cohomology of the complex $(\Lambda^\bullet\mathcal{M}, d)$, with d being the exterior derivative, and where we have used the standard notation of a superscript $\bullet$ to indicate some labelling index (i.e. $\bullet$ here covers all $0 \leq p \leq \dim \mathcal{M}$).

As with any algebraic structure, we can define a natural notion of homomorphism (i.e. "structure preserving maps") between two such structures. For two complexes, (A, d_A) and (B, d_B), such a homomorphism,

$$\varphi : (A, d_A) \to (B, d_B),$$

is called a *chain map*. Once familiar with how to construct homomorphisms, it is easy to convince ourselves that a chain map should be a vector space homomorphism

$$\varphi_i : A_i \to B_i \qquad \text{satisfying} \qquad \varphi_{i+1}d_{A,i} = d_{B,i}\varphi_i,$$

as this results in a commutative diagram

$$\begin{tikzcd} \dots \arrow[r] & A_{i-1} \arrow[r, "d_{i-1}"] \arrow[d, "\varphi_{i-1}"] & A_i \arrow[r, "d_i"] \arrow[d, "\varphi_i"] & A_{i+1} \arrow[r] \arrow[d, "\varphi_{i+1}"] & \dots \\ \dots \arrow[r] & B_{i-1} \arrow[r, "d_{i-1}"] & B_i \arrow[r, "d_i"] & B_{i+1} \arrow[r] & \dots \end{tikzcd} \tag{4.3}$$

where we have dropped the A/B subscript on our differential maps, as they should be clear from context.

Claim 4.1.1. A homomorphism between complexes gives rise to a homomorphism between cohomology groups, namely Equation (4.3) gives rise to a linear map $\alpha_i : H^i(A) \to H^i(B)$ for all $i \in \mathbb{Z}$.

Proof. Omitted. See exercise 4.4 of [6] for some good hints. □

There is one final result we need before moving on to construct our sheaf cohomology. It concerns the cases when we have a short exact sequence of complexes

$$0 \longrightarrow A_i \xrightarrow{\varphi_i} B_i \xrightarrow{\psi_i} C_i \longrightarrow 0,$$

where $\varphi : A \to B$ and $\psi : B \to C$ are chain maps, and the above short exact sequence is true for each $i \in \mathbb{Z}$. This gives us a *short exact sequence of complexes*, which we denote diagrammatically via

$$\begin{tikzcd} & & & & \\ 0 \arrow[r] & A_{i+1} \arrow[r, "\varphi_{i+1}"] \arrow[u] & B_{i+1} \arrow[r, "\psi_{i+1}"] \arrow[u] & C_{i+1} \arrow[r] \arrow[u] & 0 \\ 0 \arrow[r] & A_i \arrow[r, "\varphi_i"] \arrow[u, "d_i"] & B_i \arrow[r, "\psi_i"] \arrow[u, "d_i"] & C_i \arrow[r] \arrow[u, "d_i"] & 0 \\ 0 \arrow[r] & A_{i-1} \arrow[r, "\varphi_{i-1}"] \arrow[u, "d_{i-1}"] & B_{i-1} \arrow[r, "\psi_{i-1}"] \arrow[u, "d_{i-1}"] & C_{i-1} \arrow[r] \arrow[u, "d_{i-1}"] & 0 \\ & \arrow[u] & \arrow[u] & \arrow[u] & \end{tikzcd}$$

[3] A nice way to remember which is which is to recall that geometrically the exterior derivative gives a boundary, so something in the image of d is a coboundary.

The reason this is important is that such a short exact sequence of complexes gives rise to a long exact sequence in cohomology (the "snakeness" of the diagram is standard)

$$
\begin{tikzcd}
\arrow[r] & H^{i+1}(A) \arrow[r, "\alpha_{i+1}"] & H^{i+1}(B) \arrow[r, "\beta_{i+1}"] & \dots \\
\arrow[r] & H^i(A) \arrow[r, "\alpha_i"] & H^i(B) \arrow[r, "\beta_i"] & H^i(C) \arrow[ulll, "\delta_i"] \\
& \dots \arrow[r, "\alpha_{i-1}"] & H^{i-1}(B) \arrow[r, "\beta_{i-1}"] & H^{i-1}(C) \arrow[ulll, "\delta_{i-1}"]
\end{tikzcd}
\tag{4.4}
$$

where $\alpha_i : H^i(A) \to H^i(B)$ and $\beta_i : H^i(B) \to H^i(C)$ are the induced cohomology homomorphisms, as per Claim 4.1.1, and the $\delta_i : H^i(C) \to H^{i+1}(A)$ are known as the *coboundary operators*. The whole trick to proving this result lies in defining the coboundary operators well. This can be a lot of work to check, and so such a proof is omitted, but details can be found at the end of section 4.4 of [6].

The above constructions find use in deRham cohomology, where one splits a manifold into two, overlapping, open submanifolds and forms a short exact sequence of complexes in terms of the allowed differential forms. This then leads to a long exact sequence in deRham cohomology, so given that we know the cohomology of the submanifolds (which are normally taken to be simple things like $\mathbb{R}^d$ or S^d), we can compute the deRham cohomology of the more complicated space. This is exactly the Mayer-Vietoris sequencing we mentioned back in Section 1.2.4. For more details on this construction, see section 4.5 of [6].

4.2 Sheaf Cohomology

4.2.1 An Introduction To Germs & Sheaves: Complete Analytic Functions

In order to develop our sheaf cohomology, the first thing we need to introduce is the concept of a *topological germ*. We can give a more general definition of these using the language of functors in category theory — see [19] for more details — however this will naturally lead us too far off track. We therefore introduce germs via a particular example of *complete analytic functions*.

The basic idea of a germ is to extend the support of a function to a larger region. The obvious example is that of analytic continuation of a complex function. This is something that should be familiar from complex analysis, where we know that problems can arise. The typical example being when one analytically continues a complex logarithm, we get a multi-valued function, see Figure 4.1. We therefore need some way to deal with this mutli-valued-ness when analytically continuing. The natural thing to try is to impose some form of equivalence relation on the different branches.

> **Definition.** [Function Element] Let $f : D \to \mathbb{C}$ be an analytic function with domain D. Then we call the double (f, D) a *function element*.

So the idea to constructing our germ is to extend the domain D while keeping f well defined, i.e. single valued. First we extend the domain. Let (f_1, D_1) and (f_n, D_n) be two function elements. We say that they are equivalent if there exists a sequence of function elements with overlapping domains going from D_1 to D_n, such that the functions agree on the intersection [14]. That is $(f_1, D_1) \sim (f_n, D_n)$ iff there exists

$$
\{(f_2, D_2), ..., (f_{n-1}, D_{n-1})\} \qquad \text{such that} \qquad D_i \cap D_{i+1} \neq \emptyset \quad \text{and} \quad f_i|_{D_i \cap D_{i+1}} = f_{i+1}|_{D_i \cap D_{i+1}} \tag{4.5}
$$

where $i = 1, ..., n-1$ (so the above includes D_1 and D_n). We call the equivalence classes above *complete analytic functions* (c.a.f.), and we see that we have done nothing to ensure that the c.a.f. is not multivalued. For example, the domains in Figure 4.1 are the axes running from -5 to 5, and we see that the function is multivalued (the different spirals).

So what do we do? Well each point on the graph corresponds to two pieces of information: the point in the domain, $z \in \cup_i D_i$, and the resulting value $f(z)$. We cannot alter the later without changing the function we are considering, and so instead we must somehow alter the way we 'patch together' the different D_is. With a little thought we see that if we took the *disjoint* union, then we could distinguish $z_i \in D_i$ from $z_j \in D_j$, even if they correspond to the same point in $\cup_i D_i$.

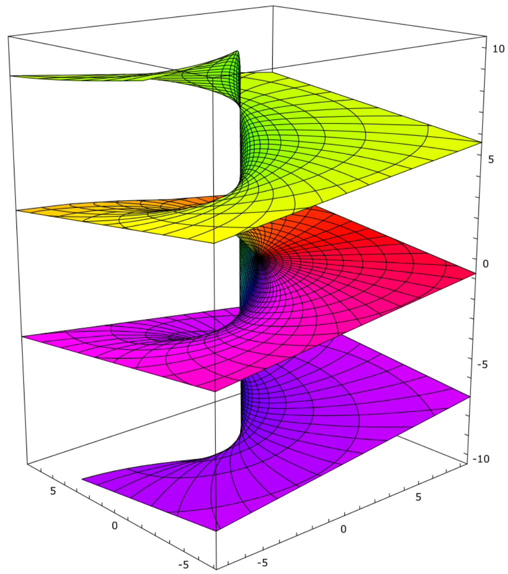


Figure 4.1: *A plot of the (imaginary part of) the analytic continuation of the complex logarithm. The colours indicate the different branches. Image from [20].*

This works great, as it allows us to separate the different 'spirals' in Figure 4.1, however we now run into a problem of over-counting. Namely, on the non-vanishing intersections $D_i \cap D_{i+1}$ we get two copies of all the points, even though f was single valued in this region. This will clearly make the resulting space, which we denote R, too big. However we now simply remember that each D_i comes with an f_i, and so we can reduce R by imposing $z_i \sim z_j$ whenever there is a neighbourhood around the point such that $f_i = f_j$ throughout this neighbourhood. Note this will only remove the over-counting as it is the *values* $f_i(z)$ and $f_j(z)$ we want to agree, and these values disagree *exactly* where f is multivalued. So in total we have a single valued, c.a.f. $f : R \to \mathbb{C}$. We call the space R the *Riemann surface* of f. Now it is clear that the analytic extension of some function element (f, D) is *not* unique, as the continuation happens *locally*. That is Equation (4.5) only depends on the local information on the intersections. We can generalise this notion then to include *all* possible analytic continuations of (f, D), which is precisely the notion of a germ.

Definition. [Germ of f at z] Let (f, D) be a function element with $z \in D$. Then the *germ of f at z* is the set of **all** function elements (f_i, D_i) such that $z \in D_i$ and there exists an open neighbourhood around z such that $f_i = f$ on this neighbourhood. We denote the germ of f at z as $[f, z]$.

To clarify, a germ of f at z is basically every possible continuation of f to some bigger domain. As we were careful to say, the germ is defined *at* z. Of course we can take some element from the germ $(f', D' := D \cup D_i)$ (i.e. a particular extension of (f, D)) and use that as a new function element to define a germ of f' at $z' \in D' \setminus D$, i.e. around a point not in the original D. In this way we continue to extend our function element, just as we did in Equation (4.5).

Definition. [Sheaf] A *sheaf* (of analytic functions on $\mathbb{C}$)[4] is the set of all germs at all points $z \in \mathbb{C}$. We denote the sheaves by $\mathcal{O}$.

Our sheaves, therefore, correspond to every possible c.a.f. on $\mathbb{C}$. By defining a map

$$\begin{aligned} \pi : \mathcal{O} &\to \mathbb{C} \\ [f, z] &\mapsto z, \end{aligned}$$

one can show that $\mathcal{O}$ is in fact a one-dimensional complex manifold [14]. The interesting point is that the *connected components* of $\mathcal{O}$ correspond to the different Riemann surfaces. This is seen simply from the fact that if (f, D_f) and (g, D_g) are two function elements whose germs, $[f, z_f]$ and $[g, z_g]$, are connected by some path $\gamma : I \to \mathcal{O}$ then, by definition of $\mathcal{O}$, we can continually deform (f, D_f) into (g, D_g). However this is the statement that $(f, D_f) \sim (g, D_g)$, and so they define the same c.a.f., and so must correspond to the same Riemann surface.

This has all been a bit abstract, so we can ground ourselves again by relating it to something we are a bit more familiar with, bundles. Going back through the definitions we see that a sheaf is essentially — we shall point out a subtle difference in a moment — a bundle where the base space is $\mathbb{C}$ and the fibre at $z \in \mathbb{C}$ are all the possible values $f(z)$. Of course we need to impose that if we take some domain $D \subseteq \mathbb{C}$ that moving through the fibres gives us our function element (f, D). Well this 'moving through the fibres' in bundle language is given by a section $s : D \to \mathcal{O}$, i.e. $(\pi \circ s)(z) = z$ for all $z \in D$. Indeed we can obtain our individual function elements in exactly this way, as per the following theorem.

Theorem 4.2.1. *Let $\mathcal{O}$ be the sheaf of analytic functions over $\mathbb{C}$. Then for any domain $D \subseteq \mathbb{C}$ there is a one-to-one correspondence between sections $s : D \to \mathcal{O}$ of $\pi : \mathcal{O} \to \mathbb{C}$ and function elements (f, D), such that $f(z) = (\sigma \circ s)(z)$ for all $z \in D$, where*

$$\begin{aligned} \sigma : \mathcal{O} &\to \mathbb{C} \\ [f, z] &\mapsto f(z). \end{aligned}$$

Proof. See page 74 of [14]. □

4.2.2 General Sheaves

We now want to generalise the notion of a sheaf outside its applications to complete analytic functions. We do this first by giving a perhaps more abstract definition via the introduction of so-called *presheaves*, as per [4]. We will then argue how this definition can be recast in a more bundle-familiar looking way, as per [14].

Definition. [Presheaf [4]] Let X be a topological space, and $U \subseteq X$ be an arbitrary open set. A *presheaf* of abelian groups,[5] $\mathcal{S}$, on X is an assignment

$$U \to \mathcal{S}(U),$$

where $\mathcal{S}(U)$ is an abelian group. This must also satisfy: given the inclusion of two open sets, $V \subseteq U \subseteq X$, there are *restriction homomorphisms*

$$r^U_V : \mathcal{S}(U) \to \mathcal{S}(V) \tag{4.6}$$

which must satisfy

$$r^V_W \circ r^U_V = r^U_W, \quad \text{and} \quad r^U_U = \mathbb{1},$$

where $W \subseteq V \subseteq U$.

If $\{U_\alpha\}$ is a collection of open sets in X, whose union we denote $U := \cup_\alpha U_\alpha$, then we denote the restriction of $s \in \mathcal{S}(U)$ to $\mathcal{S}(U_\alpha)$ by s_α, i.e.

$$\begin{aligned} r^U_{U_\alpha} : \mathcal{S}(U) &\to \mathcal{S}(U_\alpha) \\ s &\mapsto s_\alpha. \end{aligned}$$

Definition. [Sheaf [4]] A *sheaf* is a presheaf, $\mathcal{S}$, that also satisfies: if $\{U_\alpha\}$ is a collection of open sets in X with union $U := \cup_\alpha U_\alpha$, then if

(i) (Gluing): there exists a $s_\alpha \in \mathcal{S}(U_\alpha)$ and a $s_\beta \in \mathcal{S}(U_\beta)$ that agree on the overlap,

$$r^{U_\alpha}_{U_\alpha \cap U_\beta}(s_\alpha) = r^{U_\beta}_{U_\alpha \cap U_\beta}(s_\beta) \qquad \forall \alpha, \beta,$$

then there exists a $s \in \mathcal{S}(U)$ such that

$$r^U_{U_\alpha}(s) = s_\alpha \qquad \forall \alpha.$$

(ii) (Locality): there exists $s, s' \in \mathcal{S}(U)$ which satisfy

$$r^U_{U_\alpha}(s) = r^U_{U_\alpha}(s')$$

for all α, then $s = s'$.

It is hopefully clear how these defintions link to the example of c.a.f.s, in particular comparing the gluing condition to Equation (4.5), we see that the elements of $\mathcal{S}(U_\alpha)$ are the function elements over U_α, and s is the c.a.f. over U.

Although this definition is reasonably intuitive, it will prove very beneficial for us to reword it. We do this by a simple extension of Theorem 4.2.1, namely we see that $\mathcal{S}(U)$ is simply the group of sections over U. This gives us a more "bundle-looking" formalism.

Definition. [Sheaf [14]] Let X be a topological space. A *sheaf*, $\mathcal{S}$, over X is a topological space together with a projective mapping $\pi : \mathcal{S} \to X$ satisfying

(i) π is a local homeomorphism,

(ii) The stalks $\mathcal{S}_x := \text{preim}_\pi(x)$ are abelian groups,[6]

(iii) The group operations are continuous.

We have already given the explicit example of a sheaf of complete analytic functions over some complex manifold, but there are plenty of other sheaves. That is our function elements need not be analytic functions, but some other structure over X. The most common are:[7] if X is a

- Differential manifold:
  (i) $\mathcal{A}(U)$: the sheaf of C^∞ functions on U.
  (ii) $\mathcal{A}^p(U)$: the sheaf of smooth p-forms on U.
  (iii) $\mathcal{Z}^p(U)$: the sheaf of smooth, closed p-forms on U.
- Complex manifold:
  (i) $\mathcal{O}(U)$: the sheaf of holomorphic functions (i.e. our c.a.f.s).
  (ii) $\Omega^p(U)$: the sheaf of holomorphic p-forms.
  (iii) $\mathcal{A}^{p,q}$: the sheaf of smooth forms of type (p,q) — this means that the form (on twistor space, say) contains p dZ_αs and q $d\bar{Z}_\alpha$s, all wedged together.[8]
  (iv) $\mathcal{Z}^{p,q}(U)$: the sheaf of smooth, $\bar{\partial}$-closed forms of type (p,q).

4.2.3 Sheaf Homomorphisms & Sheaf Cohomology

As always, now that we have our algebraic structures, it is useful to talk about the structure preserving maps, i.e. our sheaf homomorphisms. We define these in a natural manner, i.e. if $\mathcal{S}$ and $\mathcal{T}$ are sheaves over the same topological space X, then a map

$$\varphi : \mathcal{S} \to \mathcal{T} \tag{4.7}$$

is a sheaf homomorphism if the restriction of φ to the open sets $V \subseteq U \subseteq X$ is compatible with the restriction homomorphisms. That is the following diagram commutes.

$$\begin{tikzcd} \mathcal{S}(U) \arrow[r, "\varphi_U"] \arrow[d, "r^U_V"'] & \mathcal{T}(U) \arrow[d, "r^U_V"] \\ \mathcal{S}(V) \arrow[r, "\varphi_V"'] & \mathcal{T}(V). \end{tikzcd}$$

In terms of the "bundle-like" definition, Equation (4.7) is a sheaf homomorphism if it preserves the stalks and is a group homomorphism on each stalk, i.e.

$$\varphi_x : \mathcal{S}_x \to \mathcal{T}_x \tag{4.8}$$

[7]We use the notation from [14]. Other notations exist, for example [4] uses $\mathcal{E}^p(U)$ for the sheaf of smooth p-forms on U.
[8]See exercise 9d of [14] for more info.

is a group homomorphism for all $x \in X$.

Why are sheaf homomorphisms of interest to us? Well we now note that the exterior derivative

$$d : \Lambda^p \mathcal{M} \to \Lambda^{p+1}\mathcal{M}$$

is in fact a sheaf homomorphism between $\mathcal{A}^p$ and $\mathcal{A}^{p+1}$. This gives us the exact sequence of sheaves

$$0 \xrightarrow{d} \mathcal{A}^0 \xrightarrow{d} \mathcal{A}^1 \xrightarrow{d} \dots \xrightarrow{d} \mathcal{A}^{\dim \mathcal{M}} \xrightarrow{d} 0,$$

where 0 is the zero sheaf at the corresponding level. This example makes it clear that we can construct certain cohomologies from sheaf homomorphisms. In particular, if we have a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{S} \xrightarrow{\varphi} \mathcal{T} \xrightarrow{\psi} \mathcal{R} \longrightarrow 0,$$

then we get a long exact sequence in sheaf cohomology, as per Equation (4.4).

The interesting point for us to notice is that Equation (4.8) let's us express an exact sequence of sheaves *locally*. That is, we could just as well written the short exact sequence above as

$$0_x \longrightarrow \mathcal{S}_x \xrightarrow{\varphi_x} \mathcal{T}_x \xrightarrow{\psi_x} \mathcal{R}_x \longrightarrow 0_x.$$

As we said at the beginning of this chapter, sheaf cohomology tells us how this local geometrical information translates to global information. We can see this most easily using exactly our short exact sequence example. If $U \subseteq X$ is some open neighbourhood around $x \in X$, the long exact sequence in cohomology will be of the form

$$0 \longrightarrow \mathcal{S}(U) \longrightarrow \mathcal{T}(U) \xrightarrow{\varphi} \mathcal{R}(U) \longrightarrow H^1_{\text{Sheaf}}(\mathcal{S}, U) \longrightarrow \dots,$$

where we have used that the 0-th class are exactly the sections of the sheaves, i.e. $H^0_{\text{Sheaf}}(\mathcal{S}, U) = \mathcal{S}(U)$ etc. So we see that $H^1_{\text{Sheaf}}(\mathcal{S}, U)$ is measuring the failure of the map $\varphi : \mathcal{T}(U) \to \mathcal{R}(U)$ to be surjective. That is if $H^1_{\text{Sheaf}}(\mathcal{S}, U) = 0$, then we get a short exact sequence in sections and so φ is surjective.

A standard example is to look at the short exact sequence of sheaves

$$0 \longrightarrow \mathbb{Z} \xrightarrow{i} \mathcal{O} \xrightarrow{e} \mathcal{O}^* \longrightarrow 0,$$

where $\mathbb{Z}$ is constant the sheaf of integers over X and $\mathcal{O}^*$ is the sheaf of non-vanishing holomorphic functions on X with the group operation being multiplication, and where i is an injection and $e([f, z]) := [\exp(2\pi i f), z]$. One can show that this is indeed a short exact sequence of sheaves, however if we consider the sequence of sections

$$0 \longrightarrow \mathbb{Z}(U) \longrightarrow \mathcal{O}(U) \longrightarrow \mathcal{O}^*(U) \longrightarrow 0,$$

problems can arrive at the last step, precisely because $H^1_{\text{Sheaf}}(\mathcal{O}^*, U)$ need not vanish. See page 76 of [14] for details.

4.3 Čech Cohomology

We now want to discuss the cohomology group which will prove vital for solving the z.r.m. field equations. It is known as the Čech cohomology group and is defined as follows.

Definition. [p-Simplex & p-Cochain [4]] Let X be a topological space with open cover[9] $\mathcal{U} := \{U_i\}$. Also let $\mathcal{S}$ be a sheaf on X. We call the ordered collection of $(p+1)$ open sets with non-empty intersection, $\sigma = (U_0, ..., U_p)$, a *p-simplex*. We denote the intersection, known as the *support* of σ, by

$$|\sigma| := U_0 \cap ... \cap U_p.$$

A *p-cochain* of $\mathcal{U}$ w.r.t. $\mathcal{S}$ is then a mapping

$$f : \sigma \to \mathcal{S}(|\sigma|).$$

We denote the set of p-cochains of $\mathcal{U}$ w.r.t. $\mathcal{S}$ by $C^p(\mathcal{U};\mathcal{S})$. It is easy to convince ourselves that the abelian group structure is inherited from the sheaf to $C^p(\mathcal{U};\mathcal{S})$. For simplified notational reasons, we will adopt the notation of [14], and denote the support by

$$U_{i\dots j} := |\sigma| = U_i \cap \dots \cap U_j,$$

and similarly we denote an element of the p-cochain by

$$f_{i_0\dots i_p} := f(\sigma).$$

For clarity, a p-cochain is actually a collection $\{f_{i_0\dots i_p}\}$, one for each non-empty support $|\sigma|$. However, in order to follow the terminology/notation of [14], we will just refer to $f_{i_0\dots i_p}$ as the cochain itself.

This all seems rather daunting, however it turns out that one rarely needs to consider $p > 3$ [14], and so we find some peace in that fact.

We can turn the cochain groups $C^\bullet(\mathcal{U};\mathcal{S})$ into a cochain complex by defining a *coboundary map*. This is just the differential of the complex, i.e. a mapping

$$\delta_p : C^p(\mathcal{U};\mathcal{S}) \to C^{p+1}(\mathcal{U};\mathcal{S}) \qquad \text{satisfying} \qquad \delta_{p+1}\circ\delta_p = 0.$$

This is accomplished by considering the $(p+1)$-simplex, $\sigma = (U_0, ..., U_{p+1})$, and forming a p-simplex by omitting one of the subspaces, i.e.

$$\sigma_i := (U_0, ..., \hat{U}_i, ..., U_{p+1}),$$

where the hat means "omit this". Our coboundary operator is then given by

$$\delta_p f(\sigma) = \sum_{i=0}^{p+1}(-1)^i\, r^{|\sigma_i|}_{|\sigma|} f(\sigma_i), \tag{4.9}$$

where $r^{|\sigma|}_{|\sigma_i|}$ is a restriction homomorphism, Equation (4.6).[10] We stress that the sum in Equation (4.9) is w.r.t. to the group operation inherited from the sheaves. That is it need not be addition in the usual sense, but could be multiplication. Indeed when discussing when a manifold admits a spin structure, one considers the constant sheaves $\mathbb{Z}_2$ with group operation given by multiplication. In this way we define the so-called *Stiefel-Whitney classes*. More details on this can be found in Section 11.6.3 of [8].

This looks a bit overwhelming, but if we consider the case of $p = 0$ we see that we simply have

$$\delta_0 f(U_0, U_1) = f(U_1)\big|_{U_0\cap U_1} - f(U_0)\big|_{U_0\cap U_1}.$$

If we then use the notation conventions of [14] we can write Equation (4.9) as

$$\delta_p(f_{i_0\dots i_p}) = (p+1)\rho_{[i_{p+1}} f_{i_0\dots i_p]},$$

where $\rho_i f_{j\dots k}$ meaning "restrict $f_{j\dots k}$ to $U_{ij\dots k}$", and where the factor of $(p+1)$ is included to account for the antisymmetrisation brackets.

This rewriting makes it clear that $\delta_{p+1}\circ\delta_p = 0$, as we will be antisymmetrising on two restrictions, and so the result must vanish. This completes the proof that we indeed have a complex. We then define the our p-cocycle and p-coboundary spaces

$$Z^p(\mathcal{U};\mathcal{S}) := \ker\delta_p \quad \text{and} \quad B^p(\mathcal{U};\mathcal{S}) := \mathrm{im}\delta_{p-1},$$

respectively. We then finally arrive at our pth Čech cohomology

$$\check{H}^p(\mathcal{U};\mathcal{S}) := \frac{Z^p(\mathcal{U};\mathcal{S})}{B^p(\mathcal{U};\mathcal{S})}.$$

[10] Note that $|\sigma| \subseteq |\sigma_i|$, as the former has an extra intersection.

4.3.1 Refinements

Before moving on to discuss specific examples, we should address the elephant in the room: the constructions above appear to depend *explicitly* on which open cover $\mathcal{U}$ we use. Our experiences with GR tell us that it is not a good idea to build a theory around an essentially arbitrary choice of open sets. That is if we had some other open cover $\mathcal{V}$ of X, how do we know if $\check{H}^p(\mathcal{U};\mathcal{S}) = \check{H}^p(\mathcal{V};\mathcal{S})$? The answer to this question is to introduce what is known as a *refinement*. This is basically a mapping that takes our initial open cover, say $\mathcal{U}$, to some new open cover, say $\mathcal{W}$, subject to the condition that for all W_i there exists a U_j such that $W_i \subseteq U_j$. If we denote the indexing set for $\mathcal{W}$ and $\mathcal{U}$ by I and J, respectively, then our refinement mapping is

$$r : I \to J \qquad \text{such that} \qquad W_i \subseteq U_{r(i)} \qquad \forall i \in I.$$

One can show that this induces a mapping on the cochain in the reverse direction,

$$r^* : C^p(\mathcal{U};\mathcal{S}) \to C^p(\mathcal{W},\mathcal{S}).$$

In fact one can also show that r^* commutes with the coboundary operator and so is in fact a map on the cohomology classes,

$$r^* : \check{H}^p(\mathcal{U};\mathcal{S}) \to \check{H}^p(\mathcal{W},\mathcal{S}).$$

This is an example of what is known as a *pullback*, and the reader may be familiar with such a mapping in relation to deRham cohomology; here the refinements are diffeomorphisms and the pullback is best thought of as the dual to the so-called *push forward* (which pushes vector fields in the direction of the diffeomorphism).

The claim is that if $\mathcal{W}$ is a common refinement to both $\mathcal{U}$ and $\mathcal{V}$, we can use the related mappings, $r^*_{\mathcal{U}} : \check{H}^p(\mathcal{U};\mathcal{S}) \to \check{H}^p(\mathcal{W},\mathcal{S})$ and $r^*_{\mathcal{V}} : \check{H}^p(\mathcal{V};\mathcal{S}) \to \check{H}^p(\mathcal{W},\mathcal{S})$, to map the cohomology classes to a common space. We can then compare them on this space to see if they agree. This allows us to define a new equivalence relation on the different cohomology classes; if $[f] \in \check{H}^p(\mathcal{U};\mathcal{S})$ and $[g] \in \check{H}^p(\mathcal{V};\mathcal{S})$ then $[f] \sim [g]$ if $r^*_{\mathcal{U}}[f] = r^*_{\mathcal{V}}[g]$, and so we say $\check{H}^p(\mathcal{U};\mathcal{S}) \sim \check{H}^p(\mathcal{V};\mathcal{S})$.[11] We then finally define $\check{H}^p(X;\mathcal{S})$ to be the set of such equivalence classes, thus removing the dependence of any particular open cover.

Of course this is a lot of work to check for any given Čech cohomology, and it would be much nicer to be able to use a specific open cover without the fear of open-cover-dependent results. The analogy in GR would be trying to express all of your calculations without using coordinates anywhere, or at least checking that your steps could be expressed in any equivalent coordinate system. Luckily, the following theorem exists [21].

Theorem 4.3.1 (Leray). *Let X be a topological space with sheaves $\mathcal{S}$, and let $\mathcal{U}$ be an open cover of X. Then if $\check{H}^p(U_{i_0\dots i_n};\mathcal{S}) = 0$ for all $p > 0$ and all $(i_0, ..., i_n)$, then we have the canonical isomorphism*

$$\check{H}^p(X;\mathcal{S}) \cong \check{H}^p(\mathcal{U};\mathcal{S}) \qquad \forall p \geq 0. \tag{4.10}$$

Such a cover is referred to as a ***Leray cover****.*

Proof. See page 193 of [22]. □

4.3.2 Twistor Space

We now want to consider a Čech cohomology group that will be of more direct use to us. Indeed we will see that this group will be isomorphic to the solutions of the z.r.m equations.

Recall that the subspaces

$$U_0 = \{[\pi_{0'}, \pi_{1'}] \in \mathbb{CP}^1 \,|\, \pi_{0'} \neq 0\} \quad \text{and} \quad U_1 = \{[\pi_{0'}, \pi_{1'}] \in \mathbb{CP}^1 \,|\, \pi_{1'} \neq 0\}$$

form an intersecting open cover of $\mathbb{CP}^1$. We can therefore use them to construct a Čech cohomology group, in particular we want to consider the group with sheaves given by c.a.f.s, $\check{H}^\bullet(\mathcal{U};\mathcal{O})$.

The 0-th cohomology group, $\check{H}^0(\mathcal{U};\mathcal{O})$, is simply the space of analytic functions f_i on domains U_i, subject to the cocycle constraint

$$\rho_1 f_0 = \rho_0 f_1,$$

[11] Some authors actually reserve the ˇ to indicate this equivalence class. That is they simply write the Čech cohomology classes for an open cover as $H^p(\mathcal{U};\mathcal{S})$, whereas the equivalence classes as here are $\check{H}^p(X;\mathcal{S})$.

i.e. they agree on the overlap $U_{01} := U_0 \cap U_1$. This gives us a global analytic function on $\mathbb{CP}^1$ (of homogeneity 0), and so by Liouville's theorem, it must be constant [14]:

$$\check{H}^0(\mathcal{U};\mathcal{O}) = \mathbb{C}. \tag{4.11}$$

Let's now consider the first cohomology, $\check{H}^1(\mathcal{U};\mathcal{O})$. The first thing we note is that there is no cocycle condition as we only have 2 open sets and so there is no triple intersection. This space therefore corresponds simply to functions f_{01} on U_{01}, modulo coboundaries, i.e. anything of the form $\rho_1 g_0 - \rho_0 g_1$.[12]

Thinking of U_0 and U_1 in terms of stereographic projection, we see that U_{01} is topologically an annulus (it's a sphere with the North and South poles removed). In other words, we have analytic functions defined on an annulus, but these are exactly the criteria for f_{01} to have a well defined Laurent expansion,

$$f_{01}(\xi) = \sum_{n=1}^{\infty} b_n \xi^{-n} - \sum_{n=0}^{\infty} a_n \xi^n, \qquad \text{with} \qquad \xi := \frac{\pi_{0'}}{\pi_{1'}}.$$

If we now define

$$g_0(\xi) := \sum_{n=1}^{\infty} b_n \xi^{-n} \quad \text{and} \quad g_1(\xi) := \sum_{n=0}^{\infty} a_n \xi^n,$$

and note that g_i is analytic on U_i, and so are 0-cocycles, we conclude that

$$f_{01} = \rho_1 g_0 - \rho_0 g_1 = \delta_0 g_i,$$

and so f_{01} is a coboundary. This let's us conclude

$$\check{H}^1(\mathcal{U};\mathcal{O}) = 0. \tag{4.12}$$

Arbitrary Homogeneity

As we tried to make clear above, Equations (4.11) and (4.12) are valid for analytic functions of homogeneity 0. Of course this does not exhaust the kinds of functions we want to study. So the question becomes "what if we consider sheaves of analytic functions of arbitrary homogeneity n?" We denote such sheaves by $\mathcal{O}(n)$.

First we consider the 0-th cohomology group $\check{H}^0(\mathcal{U};\mathcal{O}(n))$. We immediately note that if $n < 0$ then we are asking about global analytic functions with negative homogeneity, but these do not exist — a negative homogeneity would require a singularity somewhere, but then the function couldn't be analytic. So we conclude

$$\check{H}^0\big(\mathcal{U};\mathcal{O}(n)\big) = 0 \qquad \forall n < 0.$$

What about when $n > 0$? Well if we took n derivatives w.r.t. $\pi_{A'}$ of such a function, $g(\pi_{A'})$, we would obtain a function with homogeneity 0. That is $g(\pi_{A'})$ is a polynomial of degree n:

$$g(\pi_{A'}) = \varphi^{A_1' \dots A_n'} \pi_{A_1'} \dots \pi_{A_n'}.$$

Putting this together with Equation (4.11), we conclude

$$\check{H}^0\big(\mathcal{U};\mathcal{O}(n)\big) = \mathbb{C}^{n+1} \qquad \forall n \geq 0.$$

Next we want to consider the 1st cohomology group $\check{H}^1\big(\mathcal{U};\mathcal{O}(n)\big)$. This takes a little more work, and is easiest seen by considering the specific case $n = -1$. The space $\check{H}^1\big(\mathcal{U};\mathcal{O}(-1)\big)$ is the space of analytic functions f_{01} on U_{01} with homogeneity -1, modulo coboundaries. This is quite a complicated space. We could proceed as above, i.e. considering the Laurent expansion, however we actually want to consider a slightly more (initially) confusing approach. The reason for this will become clear next chapter.

We start by defining the 0-cochain [14]

$$h_i(\pi_{A'}) := \frac{1}{2\pi i} \int_{\Gamma_j} \frac{f_{01}(\lambda, 1) d\lambda}{\lambda \pi_{1'} - \pi_{0'}}, \qquad \text{with} \qquad \lambda := \frac{\lambda^0}{\lambda^1}, \tag{4.13}$$

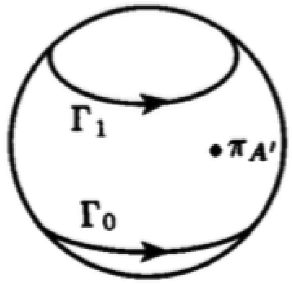


Figure 4.2: *A depiction of the contours Γ_1, Γ_2 used to define the 0-cochain Equation (4.13). Figure from [14].*

where Γ_0, Γ_1 are two contours on $\mathbb{CP}^1$ on either side of the pole $(\lambda^0, \lambda^1) = (\pi_{0'}, \pi_{1'})$, see Figure 4.2.

We note that h_i is analytic on U_i (as the pole is not contained within the contour then) and so they really are 0-cochains. Now, we pick up the pole when we consider

$$h_0(\pi_{A'}) - h_1(\pi_{A'}) = \frac{1}{2\pi i} \oint_\Gamma \frac{f_{01}(\lambda, 1)}{\lambda \pi_{1'} - \pi_{0'}} d\lambda = f_{01}(\pi_{A'}),$$

which is defined exactly on the annulus U_{01}, and so we have demonstrated that f_{01} is in fact a coboundary and so

$$\check{H}^1\big(\mathcal{U}; \mathcal{O}(-1)\big) = 0.$$

From here we can actually quickly write down the result for all $n > -1$. As before, any $f_{01}(\pi_{A'})$ with $n > -1$ can be written as a polynomial of degree n. We can then take $(n+1)$ $\pi_{A'}$ derivatives to obtain a function with homogeneity -1. We can then write this function as above, then finally multiplying by the $\pi_{A'}$ polynomial again we obtain $f_{01}(\pi_{A'})$ as a coboundary. So in total we have

$$\check{H}^1\big(\mathcal{U}; \mathcal{O}(n)\big) = 0 \qquad \forall n \geq -1.$$

Finally we just need to deal with $\check{H}^1\big(\mathcal{U}; \mathcal{O}(n)\big)$ for $n < -1$. Here we use the Laurent expansion again. We start by writing

$$f_{01}(\pi_{A'}) = \frac{1}{(\pi_{1'})^n} f_{01}(\xi, 1),$$

where again $\xi := \pi_{0'}/\pi_{1'}$. The Laurent expansion is then

$$\begin{aligned} f_{01}(\pi_{A'}) &= \left[\sum_{r=-\infty}^{-n} + \sum_{r=-n+1}^{-1} + \sum_{r=0}^{\infty} \right] \frac{a_r \xi^r}{(\pi_{1'})^n} \\ &= \sum_{r=n}^{\infty} a_{-r} \frac{(\pi_{1'})^{r-n}}{(\pi_{0'})^r} + \sum_{r=0}^{\infty} a_r \frac{(\pi_{0'})^r}{(\pi_{1'})^{n+r}} + \sum_{r=1}^{n-1} \frac{a_{-r}}{(\pi_{0'})^r (\pi_{1'})^{n-r}}. \end{aligned}$$

We note that the first two terms are well defined on U_0 and U_1, respectively, and so we see that collectively they define a coboundary. However the third term is *not* a coboundary and so in total we conclude

$$f_{01}(\pi_{A'}) \sim g_{01}(\pi_{A'}),$$

where

$$g_{01}(\pi_{A'}) := \sum_{r=1}^{n-1} \frac{a_{-r}}{(\pi_{0'})^r (\pi_{1'})^{n-r}}.$$

[12]Here we are really dealing with addition, as the group operation on $\mathcal{O}$ is addition.

That is, they are both representatives of the same equivalence class, $[f] = [g] \in \check{H}^1(\mathcal{U};\mathcal{O}(n < -1))$. Finally we note that $g_{01}(\pi_{A'})$ is defined by $(n-1)$ complex numbers (the a_{-r}s), and so

$$\check{H}^1\big(\mathcal{U};\mathcal{O}(n)\big) = \mathbb{C}^{n-1} \qquad \forall n \leq -2.$$

We summarise the above results in a table, as it will allow us to draw a very important conclusion.

n	...	-3	-2	-1	0	1	...
$\check{H}^0\big(\mathcal{U};\mathcal{O}(n)\big)$	...	0	0	0	$\mathbb{C}^1$	$\mathbb{C}^2$	...
$\check{H}^1\big(\mathcal{U};\mathcal{O}(n)\big)$	...	$\mathbb{C}^2$	$\mathbb{C}^1$	0	0	0	...

This table highlights that we have a duality between $\check{H}^0(\mathcal{U};\mathcal{O}(n))$ and $\check{H}^1(\mathcal{U};\mathcal{O}(-n-2))$. To see exactly what the duality is, consider a 0-cochain

$$g(\pi_{A'}) = \varphi^{A'_1...A'_n}\pi_{A'_1}...\pi_{A'_n} \in \check{H}^0\big(\mathcal{U};\mathcal{O}(n)\big)$$

and a 1-cochain $f(\pi_{A'}) \in \check{H}^1(\mathcal{U};\mathcal{O}(-n-2))$. Now consider the contour integral

$$(f,g) := \frac{1}{2\pi i}\oint_\Gamma g(\pi_{A'})f(\pi_{A'})\pi_{B'}d\pi^{B'},$$

where Γ is a path around the equator. This integral is complex linear, non-degenerate and depends only on the equivalence class of f [14]. It therefore defines a duality between the two spaces.[13] Next, we note that if we rewrite our contour integral as

$$(f,g) = \psi_{A'_1...A'_n}\varphi^{A'_1...A'_n},$$

with

$$\psi_{A'_1...A'_n}(\pi_{B'}) = \frac{1}{2\pi i}\oint_\Gamma \pi_{A'_1}...\pi_{A'_n}f(\pi_{B'})\pi_{C'}d\pi^{C'}, \tag{4.14}$$

we see that $\psi_{A'_1...A'_n}(\pi_{B'})$ is determined by $f(\pi_{B'})$. Finally, we note that $\psi_{A'_1...A'_n} \in \check{H}^0(\mathcal{U};\mathcal{O}(n))^*$, where $*$ denotes the vector space dual, we conclude that our duality is in fact[14]

$$\check{H}^1\big(\mathcal{U};\mathcal{O}(-n-2)\big) = \check{H}^0\big(\mathcal{U};\mathcal{O}(n)\big)^*.$$

We now claim that our cover $\mathcal{U} = \{U_0, U_1\}$ of $\mathbb{CP}^1$ is a Leray cover [14], and so, by Equation (4.10), we have actually shown

$$\boxed{\check{H}^1\big(\mathbb{CP}^1;\mathcal{O}(-n-2)\big) = \check{H}^0\big(\mathbb{CP}^1;\mathcal{O}(n)\big)^*.} \tag{4.15}$$

This is an example of a so-called *Serre duality*. It is this result which will prove invaluable when it comes to considering the solutions to the z.r.m. equations.

[13]In the same way that an inner product defines a duality between vector space and its dual.

[14]Note we have used the fact that our cohomologies have been given the structure of vector spaces, so that the vector space dual makes sense.

5 Solving The Zero Rest Mass Equations & The Penrose Transform

We are now in a good place to begin to solve the z.r.m equations, Equation (3.2). This will result in what is known as the *Penrose transform*, which basically states that [4, 14]

$$\begin{aligned}\check{H}^1\big(\mathbb{PT}^+;\mathcal{O}(-n-2)\big) &\cong \{\text{positive frequency z.r.m. fields of helicity } n\}\\ \check{H}^1\big(\mathbb{PT}^-;\mathcal{O}(-n-2)\big) &\cong \{\text{negative frequency z.r.m. fields of helicity } n\}.\end{aligned} \tag{5.1}$$

It is clear immediately that Čech cohomology, and the duality Equation (4.15), are going to play major roles here. So without further ado, let's get into it.

5.1 Integral Solutions

All the way back in Section 3.5.3 we looked at functions of twistor variables and presented the Kerr theorem, which stated that analytic g.s.f congruences arise from the intersection of a 3-dimensional plane, given by the zero set of $f(Z^\alpha)$, with $\mathbb{PN}$. We now want to do a similar thing, and demonstrate that the solutions to the z.r.m. equations arise from a function on twistor space.

Consider two *constant* dual twistors $A_\alpha = (A_A, A^{A'}) \in \mathbb{T}^*$ and $B_\alpha = (B_A, B^{A'}) \in \mathbb{T}^*$. Then consider the function

$$f(Z^\alpha) = \frac{1}{(A_\alpha Z^\alpha)(B_\beta Z^\beta)}. \tag{5.2}$$

We then impose the incidence relation, Equation (3.10), to restrict Z^α to the line L_x, where $x \in \mathbb{M}_\mathbb{C}$ is the point in complex Minkowski we which to define our z.r.m. around. That is we restrict

$$Z^\alpha = (\omega^A, \pi_{A'}) \to (ix^{AA'}\pi_{A'}, \pi_{A'}).$$

We want to use Equation (5.2) to find a solution to the z.r.m. equations. We will consider the simplest case where the field has vanishing homogeneity, and then look to generalise this to arbitrary solutions. The first thing we note is that Equation (5.2) is not projectively well defined (as A_α and B_α are *constant*), and so will cause problems on $\mathbb{CP}^1$. We therefore consider the integral

$$\varphi(x) = \frac{1}{2\pi i}\oint_\Gamma \rho_x f(Z^\alpha)\pi_{C'}d\pi^{C'} = \frac{1}{2\pi i}\oint_\Gamma f(ix^{AA'}\pi_{A'}, \pi_{A'})\pi_{C'}d\pi^{C'}, \tag{5.3}$$

where ρ_x is just the restriction to the line L_x, as indicated by the second equality. We will get a clearer picture of what Γ is below (Figure 5.1). This integral has vanishing homogeneity, and the claim is then that such a φ obeys the wave equation, $\Box\varphi = 0$. We see this by noting that

$$\frac{\partial}{\partial x^{AA'}} f(ix^{AA'}\pi_{A'}, \pi_{A'}) = i\pi_{A'}\frac{\partial}{\partial \omega^A} f(ix^{AA'}\pi_{A'}, \pi_{A'}), \tag{5.4}$$

so that

$$\frac{\partial^2}{\partial x^{AA'}\partial x^{BB'}}\varphi(x) = \frac{1}{2\pi i}\oint_\Gamma (-1)\frac{\partial^2}{\partial\omega^A\partial\omega^B}f(ix^{AA'}\pi_{A'},\pi_{A'})\pi_{A'}\pi_{B'}\pi_{C'}d\pi^{C'}.$$

If we then contract AA' with BB', using $\pi_{A'}\pi^{A'} = 0$, we get the claimed result $\Box\varphi = 0$.

We can now generalise this result to a field with helicity $\pm|s|$ by recalling Equations (3.2) and (3.3), which said that such a field would have $2s$ primed/unprimed spinor indices. Based on this index structure argument, we can propose the following integral solutions

$$\begin{aligned}\varphi_{A'_1\dots A'_{2s}}(x) &= \frac{1}{2\pi i}\oint_\Gamma \pi_{A'_1}\dots\pi_{A'_{2s}}\rho_x f(Z^\alpha)\pi_{C'}d\pi^{C'}\\ \varphi_{A_1\dots A_{2s}}(x) &= \frac{1}{2\pi i}\oint_\Gamma \rho_x\frac{\partial}{\omega^{A_1}}\dots\frac{\partial}{\partial\omega^{A_{2s}}}f(Z^\alpha)\pi_{C'}d\pi^{C'}.\end{aligned} \tag{5.5}$$

where homogeneity arguments (i.e. $\varphi_{\dots}$ must have vanishing homogeneity) tell us we require that the first $f(Z^\alpha)$ have weight $-s-2$, while the second has $s-2$. To show that these are indeed solutions we simply use Equation (5.4).

This is great, but we have actually been a little careless. To see why we go back to Equation (5.2) and note that restricting to L_x makes our contractions

$$\begin{aligned}A_\alpha Z^\alpha &= (iA_Ax^{AA'} + A')\pi_{A'} =: \alpha^{A'}\pi_{A'}\\ B_\alpha Z^\alpha &= (iB_Ax^{AA'} + B')\pi_{A'} =: \beta^{A'}\pi_{A'},\end{aligned} \tag{5.6}$$

where we have defined $\alpha^{A'}$ and $\beta^{A'}$. Our integral, Equation (5.3), then becomes

$$\varphi(x) = \frac{1}{2\pi i}\oint_\Gamma \frac{1}{(\alpha^{A'}\pi_{A'})(\beta^{B'}\pi_{B'})}\pi_{C'}d\pi^{C'}. \tag{5.7}$$

We now note that this is only well defined if the two poles are distinct, i.e. if $\alpha^{A'} \neq \beta^{A'}$ along L_x. Since $\alpha^{A'}$ and $\beta^{A'}$ are distinct, we can use the lowered index versions as a coordinate basis for $\pi_{A'}$:

$$\pi_{A'} = \alpha_{A'} + z\beta_{A'}.$$

Substituting this into the integral above, using $\alpha^{A'}\alpha_{A'} = \beta^{A'}\beta_{A'} = 0$ and the fact that they are constants (so we only get dz), we have

$$\varphi(x) = \frac{1}{2\pi i}\oint_\Gamma \frac{1}{(\alpha^{A'}\beta_{A'})z}dz = \frac{1}{\alpha^{A'}\beta_{A'}}.$$

Now comes the important observation: if our two dual twistors meet in some other line, L_y, then there we must have $\alpha^{A'}\beta_{A'} = 0$. Putting this together with the definitions in Equation (5.6), we conclude that

$$\alpha^{A'}\beta_{A'} = \frac{1}{2}A_AB^A(x-y)^2 \qquad \Longrightarrow \qquad \varphi(x) = \frac{2}{A_AB^A(x-y)^2},$$

where y is considered as some fixed point. We can verify that this is indeed a solution to the wave equation, however we have the immediate problem that all the x that are null separated from y give a singular behaviour for our field! If $\varphi(x)$ is going to be our field, then, we must restrict the domain. This restriction allows us to give a geometrical picture of positive/negative frequency fields [14]:

(i) If we take L_y to lie entirely in $\mathbb{PT}^-$, then we can take the domain to be all of $\mathbb{PT}^+$, i.e. any $L_x \subset \mathbb{PT}^+$ gives a non-singular $\varphi(x)$. This domain in $\mathbb{M}_\mathbb{C}$ corresponds to all time-like future pointing vectors, so the domain is $\mathbb{M}_\mathbb{C}^+$, known as the *future tube*.

(ii) Similarly if we take L_y to lie entirely in $\mathbb{PT}^+$ then we get $L_x \subset \mathbb{PT}^-$ and the *past tube* $\mathbb{M}_\mathbb{C}^-$.

These ideas clearly carry over to the higher helicity fields, and with a bit of faith we can begin to see where Equation (5.1) comes from: Equation (5.5) tells us that we want homogeneous functions of degree $-n-2$, while the conditions above tell us that the domains give us positive and negative frequency fields. This idea is strengthened when we compare Equations (4.14) and (5.5): the solutions to our z.r.m. equations form a Serre duality, and we should think of our twistor functions as elements of the cohomology class $\check{H}^1(\mathcal{U};\mathcal{O}(-n-2))$ where $\mathcal{U}$ is an open cover of $\mathbb{PT}^+$.

Before moving on to show this isomorphism in more detail we want to point out an important property of our functions $f(Z^\alpha)$. We recall that (in the case of vanishing helicity field, the extension should be clear) we required our two poles to be distinct. This was just the statement that, for a positive frequency field, the β-planes defined by the dual twistors A_α and B_α do not intersect in $\mathbb{PT}^+$. The intersection of L_x with these two planes then gives the two poles on our Riemann sphere and the contour runs between these two poles, see Figure 5.1. This contour corresponds to a plane in $\mathbb{PT}$ lying between the two β-planes, and moving the contour towards the poles corresponds to moving this plane towards the corresponding β-plane.

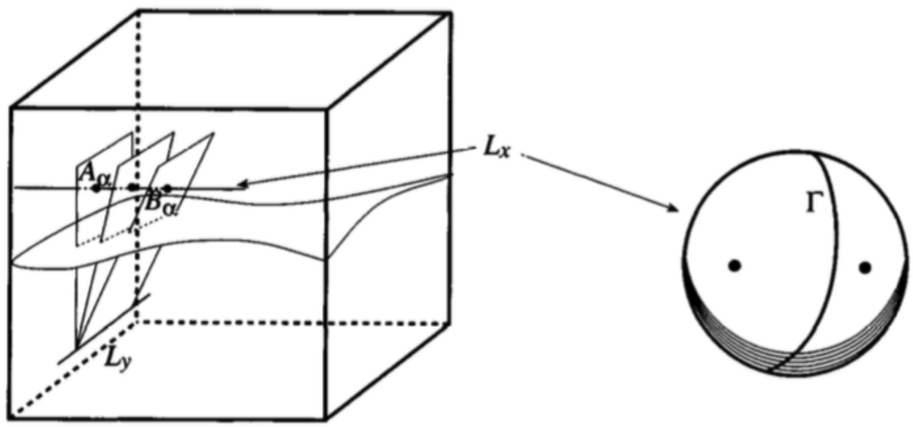


Figure 5.1: *A positive frequency field with vanishing helicity, $\varphi(x)$, is defined by the integral Equation (5.7), where the β-planes corresponding to the dual twistors A_α and B_α meet at $L_y \subset \mathbb{PT}^-$ and do not intersect in $\mathbb{PT}^+$. The intersection of $L_x \subset \mathbb{PT}^+$ with these β-planes gives two distinct poles on the Riemann sphere. The contour, Γ, for our integral then corresponds to a 'moveable' plane lying between the two β-planes; when this plane overlaps with a β-plane, the corresponding pole is picked up. Figure from [14].*

As the two poles lie on different sides of Γ, it is clear that if we add to $f(Z^\alpha)$ a function, $h(Z^\alpha)$, which is holomorphic on one side of this contour (even if it is singular on the other side), we still get the same field $\varphi(x)$ as we can simply contract Γ to the holomorphic side. As our integrals are linear, we can actually add two such functions, which are singular on opposite sides of Γ. In this way we have the total freedom

$$f \mapsto f + h - \hat{h},$$

where $h, \hat{h}$ are holomorphic on opposite sides of Γ. In terms of cohomology, this shift can be encoded in changing the representative $[f] \in \check{H}^1(\mathcal{U};\mathcal{O}(-2))$ by a coboundary, i.e. $h, \hat{h} \in B^1(\mathcal{U};\mathcal{O}(-2))$ [14].

5.2 Proving The Penrose Transform

The above arguments were simply suggestions that the Penrose transform holds, however they do not constitute a rigorous proof in themselves. The aim of this section is to provide a more satisfying proof. In particular, we want to show it holds for the cases of non-vanishing helicity.

First we note what the Penrose transform is actually doing. In order to do this we need to recall our double fibration picture, Equation (3.14), which we depict again now with more standard notation:

$$\begin{tikzcd} & \mathbb{F} \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ \mathbb{PT} & & \mathbb{M}_{\mathbb{C}} \end{tikzcd}$$

$\mathbb{F}$ is a primed spinor bundle and the maps are given as before, i.e.

$$\mu : (x^{AA'}, \pi_{A'}) \mapsto (ix^{AB'}\pi_{B'}, \pi_{A'}), \quad \text{and} \quad \nu : (x^{AA'}, \pi_{A'}) \mapsto x^{AA'}.$$

Now, clearly the left-hand side of the Penrose transform, $\check{H}^1(\mathbb{PT}^{\pm}; \mathcal{O}(-n-2))$, is a cohomology class on $\mathbb{PT}$. However, the right-hand side are sections of the sheaf of analytic massless free-fields of helicity n on $\mathbb{M}_{\mathbb{C}}^{\pm}$. These two objects live on different spaces, and so it is really not a trivial task to compare the two. This is exactly where the double fibration comes in: we can use $\mathbb{F}$ as an intermediate stepping stone to translate information from $\mathbb{PT}$ to information on $\mathbb{M}_{\mathbb{C}}$. That is we can try use the mappings

$$\nu \circ \mu^{-1} : \mathbb{PT} \to \mathbb{M}_{\mathbb{C}} \quad \text{and} \quad \mu \circ \nu^{-1} : \mathbb{M}_{\mathbb{C}} \to \mathbb{PT}$$

to translate data between the two spaces. This is exactly how we shall prove the Penrose theorem, however we will naturally hide a lot of the technical steps, as these would take up too much space to show completely. A much more complete proof[1] can be found in section 7.2 of [4]. Here we will follow the more condensed proof given in [14], however we first make an important comment.

Our starting point is a Čech cohomology class on an open subset $U \subset \mathbb{PT}$. We argued above that we wanted to consider specifically the cases of $\mathbb{PT}^{\pm}$, and although those arguments were rather clear, we can now give further support for these choices. As we touched upon in Section 4.3.1, the natural way to lift a cohomology structure from one space to another is via a pullback. Consider the open subspaces $U \subset \mathbb{PT}$ and $U' \subset \mathbb{F}$, related via $U = \mu(U')$. Then we want to consider the particular case of

$$\mu^* : \check{H}^1(U; \mathcal{O}) \to \check{H}^1(U'; \mu^{-1}\mathcal{O}),$$

where $\mu^{-1}\mathcal{O}$ is a pullback[2] of the sheaves to U', which is a well-defined structure (see section 7.1 of [4]). Now as the Penrose transform is an isomorphism, we naturally ask the question "when is such a pullback an isomorphism?" The answer turns out to be topological, namely we require $\mu : U' \to U$ to be so-called *elementary*. Similar topological restrictions apply when we then try and push this data down from $\mathbb{F}$ to $\mathbb{M}_{\mathbb{C}}$. It turns out that taking $U = \mathbb{PT}^{\pm}$ will satisfy these topological restrictions [4], and so we the Penrose transform seems at least plausible.

5.2.1 Positive Helicity

We start by considering the simpler cases of positive helicity. We define, [14], $\mathcal{Z}'_n(m)$ to be the sheaf of germs of symmetric n-index primed spinor fields, $\varphi_{A'\dots B'}(x, \pi)$, holomorphic on $\mathbb{F}^+$, homogeneous with degree m in $\pi_{A'}$, which are also z.r.m. fields, i.e.

$$\nabla_A{}^{A'} \varphi_{A'\dots B'} = 0.$$

Now the idea to proving the Penrose transform will be to construct a short exact sequence of these sheaves, and then 'pluck it out' of the corresponding long exact sequence in cohomology. We therefore need to make the above into a complex, which we do by considering the surjective mapping [14]

$$\pi^{A'} : \mathcal{Z}'_{n+1}(m-1) \to \mathcal{Z}'_n(m)$$
$$\varphi_{A'B'\dots C'} \mapsto \pi^{A'}\varphi_{A'B'\dots C'}.$$

[1] This also uses the language of *resolutions*, details of which can be found in the reference, of course.

[2] To those familiar with pullbacks, the idea of denoting a pullback with an inverse symbol, rather then the standard superscript $*$, seems strange. This is done here to avoid confusion with the map μ^* we are trying to define, and the specific $\mu^{-1}\mathcal{O}$ notation is used to match [4].

As this map is surjective, if we consider its kernel we can obtain a short exact sequence. That is we define

$$\mathcal{T} := \{\psi_{A'...B'} \in \mathcal{Z}'_{n+1}(m-1) \,|\, \pi^{A'}\psi_{A'...B'} = 0\},$$

then we get a short exact sequence

$$0 \longrightarrow \mathcal{T} \xrightarrow{\iota} \mathcal{Z}'_{n+1}(m-1) \xrightarrow{\pi^{A'}} \mathcal{Z}'_n(m) \longrightarrow 0,$$

where ι is an inclusion mapping. We can actually construct $\mathcal{T}$ explicitly: recall that symmetric spinors can be factorised into their principal null directions, Equation (2.19). Well, by definition $\psi_{A'...B'} \in \mathcal{T}$ must satisfy

$$\pi^{A'}\psi_{A'...B'} = 0 \qquad \Longrightarrow \qquad \psi_{A'...B'} \propto \pi_{A'}.$$

If we then impose the symmetry condition we see that the p.n.d.s of $\psi_{A'...B'}$ are simply the $\pi^{A'}$s, i.e.

$$\psi_{A'...B'} = \pi_{A'}...\pi_{B'}f(x,\pi),$$

where $f(x,\pi)$ is the proportionality function.

Next it follows from the definition of the mapping that $\psi_{A'...B'}$ must be homogeneous of degree $m-2$ in $\pi_{A'}$, and so we conclude that $f(x,\pi)$ must be homogeneous of degree $m-n-2$, to account for the n $\pi_{A'}$ factors in $\psi_{A'...B'}$. We define $\mathcal{T}(m-n-2)$ to be the set of such $f(x,\pi)$s. We can then define an injection

$$\begin{aligned}\iota' : \mathcal{T}(m-n-2) &\to \mathcal{Z}'_{n+1}(m-1)\\ f(x,\pi) &\mapsto \pi_{A'}...\pi_{B'}f(x,\pi),\end{aligned}$$

and our short exact sequence becomes

$$0 \longrightarrow \mathcal{T}(m-n-2) \xrightarrow{\iota'} \mathcal{Z}'_{n+1}(m-1) \xrightarrow{\pi^{A'}} \mathcal{Z}'_n(m) \longrightarrow 0.$$

Let's consider the specific case of $m=0$, then the corresponding long exact sequence in cohomology contains the piece

$$\cdots \to \check{H}^0\big(\mathbb{F}^+;\mathcal{Z}'_{n+1}(-1)\big) \to \check{H}^0\big(\mathbb{F}^+;\mathcal{Z}'_n(0)\big) \to \check{H}^1\big(\mathbb{F}^+;\mathcal{T}(-n-2)\big) \to \check{H}^1\big(\mathbb{F}^+;\mathcal{Z}'_{n+1}(-1)\big) \to \dots$$

Recalling that the zeroth Čech cohomology just corresponds to global sections, we see that the first term must vanish as, for a fixed $x^{AA'}$, it would correspond to global sections on $\mathbb{CP}^1$ with homogeneity -1, which we already showed must vanish. Equally it follows from the fact that our map $\mu : \mathbb{F} \to \mathbb{PT}$ is elementary that the final term vanishes (see Lemma 7.2.1 of [4]). Exactness then leaves us to conclude that

$$\check{H}^0\big(\mathbb{F}^+;\mathcal{Z}'_n(0)\big) \cong \check{H}^1\big(\mathbb{F}^+;\mathcal{T}(-n-2)\big).$$

Now the left-hand side is the cohomology of solutions to the z.r.m. field equations on $\mathbb{F}^+$ with vanishing homogeneity in $\pi_{A'}$. The fact that it is defined on $\mathbb{F}^+$ tells us that the fields have positive frequency, i.e. the spacetime points are $\mathbb{M}^+_{\mathbb{C}}$. So these are exactly the right-hand side of our Penrose transform Equation (5.1).

Finally the term on the right-hand side is a cohomology class of twistor functions on $\mathbb{F}^+$. However, as they are twistor functions, we can simply push them down to $\mathbb{PT}^+$ without changing anything. That is the fact that $\psi_{A'...B'}$ satisfies the z.r.m. equations tells us that

$$\pi^{A'}\nabla_{AA'}f(x,\pi) = 0,$$

and so it is constant on the α-planes and so can freely be pushed down onto $\mathbb{PT}^+$. The sheaves are then analytic functions of degree $(-n-2)$, and so we have

$$\check{H}^1\big(\mathbb{F}^+;\mathcal{T}(-n-2)\big) \cong \check{H}^1\big(\mathbb{PT}^+;\mathcal{O}(-n-2)\big),$$

which if we put together with the above gives us exactly the Penrose transform. A similar argument can be made for $\mathbb{PT}^-$ and $\mathbb{M}^-_{\mathbb{C}}$,[3] and so we can summarise the positive helicity case as [4]

[3]Indeed you can show the isomorphism holds for a general open subspace, subject to the topological descriptions mentioned above. See Theorem 7.2.3 of [4] for the general statement.

$$\check{H}^1\big(\mathbb{PT}^{\pm};\mathcal{O}(-n-2)\big) \cong \Gamma(\mathbb{M}_{\mathbb{C}}^{\pm};\mathcal{Z}'_n) \qquad \forall n \geq 1.$$

The explicit inverse coboundary operator $\delta_0^{-1} : \check{H}^1\big(\mathbb{PT}^{+};\mathcal{O}(-n-2)\big) \to \Gamma(\mathbb{M}_{\mathbb{C}}^{+};\mathcal{Z}'_n)$ is constructed on pages 93-94 of [14], and the interested reader is directed there.

5.2.2 Negative Helicity

We have demonstrated the positive helicity case ($n \geq 1$), so we now turn our attention to the negative helicity cases. That is we want to show that

$$\check{H}^1\big(\mathbb{PT}^{\pm};\mathcal{O}(n-2)\big) \cong \Gamma(\mathbb{M}_{\mathbb{C}}^{\pm};\mathcal{Z}'_{-n}) \qquad \forall n \geq 1.$$

Unfortunately this is significantly more involved, and a complete treatment (as in [4]) involves introducing structures like *Stein manifolds*. As before, we will gloss over these details and follow the proof given in [14]. However, even with this more 'streamlined' proof, we still need to introduce potentials.

Again the idea is to use a short exact sequence of sheaves and then pluck the result from a long exact sequence in cohomology. This time we consider the short exact sequence of sheaves on $\mathbb{F}^+$

$$0 \longrightarrow \mathcal{T}(n) \longrightarrow \mathcal{K}(n) \xrightarrow{D_A} \mathcal{Q}_A(n+1) \longrightarrow 0,$$

where

- $\mathcal{K}(n)$ is the sheaf of holomorphic functions with homogeneous degree n in $\pi_{A'}$.
- D_A is the operator, $D_A := \pi^{A'}\nabla_{AA'}$.
- $\mathcal{T}(n)$ is the kernel of D_A, so again represents twistor functions of homogeneity n.
- $\mathcal{Q}_A(n+1)$ is the sheaf of spinor-valued functions, $\psi_A(x,\pi)$, of homogeneity $(n+1)$ in $\pi_{A'}$, satisfying $D^A\psi_A = 0$.

It is clear that $\mathcal{T}(n)$ can be mapped injectively to $\mathcal{K}(n)$, and the claim is that $D_A : \mathcal{K}(n) \to \mathcal{Q}_A(n+1)$ is surjective [14], and so the above sequence is indeed exact.

We now look at the piece of the corresponding long exact sequence

$$0 \to \check{H}^0\big(\mathbb{F}^+;\mathcal{T}(n)\big) \to \check{H}^0\big(\mathbb{F}^+;\mathcal{K}(n)\big) \to \check{H}^0\big(\mathbb{F}^+;\mathcal{Q}_A(n+1)\big) \to \check{H}^1\big(\mathbb{F}^+;\mathcal{T}(n)\big) \to \check{H}^1\big(\mathbb{F}^+;\mathcal{K}(n)\big) \to \dots$$

We shall describe what each of these cohomology classes are in just a moment, however first we note that, for exactly the same reasons as above, we can push $\check{H}^{0/1}(\mathbb{F}^+;\mathcal{T}(n))$ down to $\mathbb{PT}^+$ and obtain $\check{H}^{0/1}(\mathbb{PT}^+;\mathcal{O}(n))$. We then also claim that, again by Lemma 7.2.1 of [4], that $\check{H}^1(\mathbb{F}^+;\mathcal{K}(n)) = 0$. So what are cohomology groups? Well $\check{H}^1(\mathbb{F}^+;\mathcal{T}(n)) \cong \check{H}^1(\mathbb{PT}^+;\mathcal{O}(n))$ is the space we want, and the other three are:

- $\check{H}^0(\mathbb{F}^+;\mathcal{T}(n)) \cong \check{H}^0(\mathbb{PT}^+;\mathcal{O}(n))$: This is the space of global twistor functions of homogeneity n. These are order n polynomials $\mu = \pi^{A'}...\pi^{B'}\mu_{A'...B'}$ lying in the kernel of D_A, which implies

  $$\pi^{C'}\nabla_{CC'}\mu = \pi^{A'}...\pi^{B'}\pi^{C'}\nabla_{C(C'}\mu_{A'...B')} = 0.$$

  We shall denote this space as T_n in what follows.

- $\check{H}^0(\mathbb{F}^+;\mathcal{K}(n))$: these are the same weight n polynomials as above, however they need not lie in the kernel of D_A. We shall denote this space Λ_n in what follows.

- $\check{H}^0(\mathbb{F}^+;\mathcal{Q}_A(n+1))$: This is the space of spinor fields with $(n+1)$ primed indices, $\psi_A = \psi_{AA'...C'}\pi^{A'}...\pi^{C'}$, which lie in the kernel of D^A, i.e.

  $$D^A\psi_A = \pi^{A'}...\pi^{C'}\pi^{D'}\nabla_{(D'}{}^{A}\psi_{A'...C')A} = 0.$$

  We shall denote this space by Ψ_n.

So our long exact sequence becomes

$$0 \longrightarrow T_n \xrightarrow{\iota} \Lambda_n \xrightarrow{\sigma} \Psi_n \xrightarrow{\delta_0} \check{H}^1\big(\mathbb{PT}^+; \mathcal{O}(n)\big) \longrightarrow 0, \tag{5.8}$$

where ι is again an inclusion mapping, $\sigma : \lambda_{A'\dots B'} \mapsto \nabla_{A(A'}\lambda_{B'\dots C')}$, and δ_0 is the coboundary operator, as per Equation (4.4). Again the explicit construction of this mapping can be found on pages 96-98 of [14].

This looks nice, however it doesn't seem quite as nice as the positive helicity case where we could easily extract the isomorphism from the long exact sequence itself. So what do we do here? The answer is we need to introduce potentials modulo gauge transformations [14].

Consider a field $\psi_A \in \Psi_n$, i.e. a field $\psi_{AA'\dots C'}$ such that $\nabla^A{}_{(D'}\psi_{AA'\dots C')A} = 0$. Then define a new field

$$\varphi_{AB\dots D} := \nabla^{B'}{}_{(B}\dots\nabla^{D'}{}_{D}\psi_{A)B'\dots D'}. \tag{5.9}$$

This field has $(n+2)$ unprimmed spinor indices and it follows from the conditions above that it is a solution of the z.r.m. equations, i.e. $\nabla_{A'}{}^{A}\varphi_{AB\dots D} = 0$. This field therefore corresponds to a helicity $-\frac{1}{2}(n+2)$ z.r.m. field, as per Equation (3.2). We shall denote the space of such fields by Φ_{n+2}.

This is the space we want to be isomorphic to $\check{H}^1(\mathbb{PT}^+; \mathcal{O}(n))$, and so we somehow want to show that we have the same long exact sequence as Equation (5.8) but with $\check{H}^1(\mathbb{PT}^+; \mathcal{O}(n))$ replaced with Φ_{n+2}. How do we do that? Well we note that the transformation

$$\psi_{AA'\dots C'} \to \psi_{AA'\dots C'} + \nabla_{A(A'}\lambda_{B'\dots C')}$$

will still lie in the kernel of D^A. It is therefore a gauge transformation on the space Ψ_n. We then note that if $\lambda_{B'\dots C'} \in T_n$ then it lies in the kernel of D_A and so leaves $\psi_{AA'\dots C'}$ itself unchanged. This then gives us an exact sequence

$$0 \longrightarrow T_n \xrightarrow{\iota} \Lambda_n \xrightarrow{\sigma} \Psi_n \xrightarrow{\nu} \Phi_{n+2} \longrightarrow 0,$$

where $\nu : \psi_{AB'\dots D'} \mapsto \varphi_{AB\dots D}$ via Equation (5.9), which we claim is indeed surjective (as needed for exactness). Finally we simply compare our two exact sequences and conclude (making the argument that a similar calculation holds for $\mathbb{PT}^-$ etc)

$$\boxed{\check{H}^1\big(\mathbb{PT}^\pm; \mathcal{O}(n-2)\big) \cong \Phi_n = \Gamma(\mathbb{M}^\pm_{\mathbb{C}}; \mathcal{Z}'_{-n}) \qquad \forall n \geq 1,}$$

which was the desired result.

Example: Maxwell's Fields

This construction might seem a bit abstract, but we can see that it is actually rather straight forward by considering the example of a Maxwell field. Recall that this corresponds to a 2 index unprimmed z.r.m. field, Equation (3.1), which we now denote φ_{AB}. The potential field is an element of Ψ_0, i.e. $\psi_{AA'}$ such that $\nabla^A{}_{(A'}\psi_{B')A} = 0$. Now T_0 is simply $\mathbb{C}$, i.e. it is the set of constant functions, and so the gauge transformation is simply

$$\psi_{AA'} \to \psi_{AA'} + \nabla_{AA'}\Lambda,$$

where $\Lambda \in \mathbb{C}$, and where the notation is used to make comparison to standard QFT literature straight forward.

6 | Conclusion & Further Reading

We present a reasonably detailed introductory review of twistor theory, discussing the main concepts and presenting some of the significant results and their relations to physics.

We show how twistor theory can build on the already well understood double covering map, which relates spinors and vectors. Just as this map allows us to view spinors as more 'primitive' than vectors, twistor theory allows us to view twistor space as more elementary than the points in spacetime themselves. This relation is captured in the incidence relation, Equation (3.10), and, by studying its geometrical properties, we show that:

(i) A point $x \in \mathbb{M}_{\mathbb{C}}$ in complexified Minkowski spacetime corresponds to a line $L_x \in \mathbb{PT}$ in projective twistor space. See Section 3.3.1.

(ii) The intersection of two lines $L_x, L_y \in \mathbb{PT}$, which represents a point in $\mathbb{PT}$, corresponds to the associated points $x, y \in \mathbb{M}_{\mathbb{C}}$ being null separated. See Section 3.3.2.

Relation (ii) allows us to introduce α-planes and β-planes, which are totally null 2-planes in $\mathbb{M}_{\mathbb{C}}$ who's tangent bivectors are self-dual and anti-self-dual, respectively. This gives us a nice pictorial description of the twistor correspondence, Figure 3.1. From here we show that if an α-plane contains a real point, and so corresponds to the real Lorentzian Minkowski spacetime, $\mathbb{M}$, that the corresponding twistor is null. The reverse is also true: a null twistor gives a point in $\mathbb{M}$. In this way we show that the space of null twistors, $\mathbb{PN}$, corresponds to the Lorentzian signature hypersurface of $\mathbb{M}_{\mathbb{C}}$.

We show in Section 3.4.2 that twistor space is conformally invariant, and so can only encode the conformal *class* of the spacetime itself. We remedy this by introducing the infinity twistor, and we go through the explicit example for Minkowski spacetime.

By considering geodesic congruences and introducing the notion of complex shear, we present the Kerr theorem, which states that all geodesic shear free congruences arise from the intersection of the zero set of a twistor function with a hypersurface in projective twistor space. We give the important example of the Robinson congruence, which is where twistor theory gets its name.

The Kerr theorem forms the basis for our attempt to solve the zero-rest-mass field equations in terms of twistor functions. We solve this problem by first introducing the ideas of sheaf cohomology in Chapter 4, and then present the Penrose transform in the proceeding chapter. This tell us that there is an isomorphism between the solutions of the z.r.m. field equations with helicity n and the first Čech cohomology group $\check{H}^1(\mathbb{PT}^{\pm}; \mathcal{O}(-n-2))$, Equation (5.1). This is a very significant result, as the former is obviously very physical, while the later is an incredibly precise mathematical construction.

This project by no means exhausts the vast shadow cast by the umbrella of twistor theory's ideas. Indeed what we have actually presented is the *linear* Penrose transform. There is a non-linear extension to this known as the (Penrose-)Ward transform. This was first presented in 1976 in Penrose's paper [23], which discusses how to produce *single* gravitons from a curved twistor space. The curvature is actually split into right-/left-handed, which can be related to whether the spacetime is self-dual or anti-self dual. A so-called *right-flat* (so left-curved) space gives anti-self dual solutions, and this paper shows how such a construction gives rise to a general left-handed graviton state. The Ward transform allows us to study gauge theory in more detail, and the transform relates anti-self-dual Yang-Mills (ASDYM) solutions to holomorphic vector bundles on projective twistor space [24]. This correspondence allows one to then study instantons, i.e. ASDYM solutions with finite action, i.e.

$$\int \mathrm{Tr}(F \wedge \star F) < \infty.$$

As we said the Ward transform only encodes information about *anti*-self-dual solutions. Indeed it is one of the infamous problems in twistor theory known as the *Googly Problem*,[1] which states that we don't know how to encode self-dual solutions in the geometry of $\mathbb{PT}$.

A discussion of these topics, as well a plethora of others, can be found in the Twistor Newsletters [25, 26].

These Newsletters cover information up to the turn the millenia, and so do not contain information about Witten's work [3] mentioned in the introduction. This is an extension of the Parke-Taylor formula, which uses the fact that a null vector corresponds to an outer product of two spinors with opposite chirality (see Equation (2.7)) to compute tree level gluon amplitudes [27]. This construction was first related to twistor theory in 1988 by Nair's extension in [28], which included $\mathcal{N} = 4$ supersymmetry and expressed it as an integral in twistor space. Witten's idea was to then extend this by considering a string theory with target space given by super-twistor space. This formalism gave a formulation for $\mathcal{N} = 4$ super-Yang-Mills theory.

[1] This problem gets its name from the bowling technique in cricket... who would have thought!

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