TORIC GEOMETRY

NOTES PRODUCED BY

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COMMENTS & ACKNOWLEDGEMENTS

The primary goal of these notes is to aid in/test my understanding of the work I am doing for my PhD at Durham university. For this reason they are highly tailored to my own thought process, which will obviously have an effected on the level of detail both present and missing on the content that follows. That said, I do like to write my notes with a "imagine teaching this to someone" mentality, as it forces me to really think about the material rather than blindly copy it. For this reason (and because I'm producing the notes anyway) I am making these notes available on my website. For updated versions of these notes + all other stuff I've done, click the following link:

Below I list the main sources used in producing these notes, and a more complete list of references can be found at the end of the document.

- *Mirror symmetry. Vol. 1: K. Hori et al*:
  A lot of the work presented in these notes is based around my reading of chapter 7 of this book. It presents the material in a very neat manner, with lots of reoccurring examples to help build up a solid understanding.

- *Toric varieties: D. Cox, J. Little, & H. Schenck:*
  This is a very nice book with lots of mathematical rigour. It proved particularly useful in my understanding of the polytope constructions & Hizerbruch surfaces.

- *K3 surfaces and string duality: P. Aspinwall:*
  This is a very nicely presented review of K3 surfaces. It discusses a lot of material that is not discussed explicitly in these notes, but definitely provides excellent reference material for some of the more subtle points.[1]

Of course I would also like to extend a huge thank you to my supervisor Dr. Andreas Braun for suggesting and discussing the work with me. A lot of this material, particularly the examples, came directly from my discussions with him.

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[1]In particular, the relation between subdiving a fan and deformations of the defining equation when discussing blow ups.

Contents

0 | Introduction

These notes are meant to build on the Complex Manifold notes previously written. The main purpose of those notes was to introduce Calabi-Yau manifolds and present a method of obtaining them as hypersurfaces in complex projective spaces. Although this construction was quite adaptable, i.e. we can consider direct sums of $\mathbb{CP}^n$s etc, however it was a reasonable amount of work to do calculations this way. In particular, we showed at the end of those notes that the hyperplane class of *weighted* complex projective spaces was fractional. We mentioned that this corresponded to *orbifold* singularities, but didn't comment more on how to do with these. The content of those notes is based around the mathematical area of *differential* geometry.

The idea of these notes is to introduce a new mathematical tool that makes questions of "is the hyperplane class fractional?" and "how do we deal with orbifold singularities?" essentially trivial combinatorics games. This mathematical tool is *algebraic* geometry, and in particular *toric* geometry.

The content is laid out as follows:

- Chapter 1 briefly introduces the main preliminary mathematics that it's reasonable to not already be familiar with.

- Chapter 2 then introduces a key structure, *divisors*. These are basically formal objects that encode hypersurfaces in some ambient space. As we will see, when we talked about hyperplane bundles in the Complex Manifolds notes, what we were really talking about was a line bundle associated to a divisor, hence the name *hyperplane* bundle.

- Chapter 3 forms the main content. It is here we really discuss toric geometry, obviously introducing all the relevant definitions. We give the procedure of how to construct a toric variety from a fan using the 'homogeneous coordinate' approach. We then go on to demonstrate the real immense power of this construction by the fact that such a simple combinatorics game contains so much information about the resulting space. We, of course, discuss how to construct Calabi-Yau manifolds in these spaces. We introduce some examples that we keep returning to in order to help keep the material grounded.

- Chapter 4 is a short introduction and discussion of so-called *elliptic fibrations*. This chapter really demonstrates how useful the toric techniques are to constructing highly non-trivial spaces.

- Chapter 5 is then a quick summary of the material.

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1 | Preliminaries

We want to build on the work done in the Complex Manifold notes, by discussing toric geometry as a method of resolving the orbifold singularities present in weighted projective spaces. Indeed, we will see that toric geometry actually gives us a really neat way to produce Calabi-Yau manifolds with all kinds of desired properties. In order to do all this, however, we first need some preliminaries, which this chapter shall cover. We will only introduce the bare minimum, and present a lot of this as a list of definitions that we can refer back to when needed.

1.1 Orbifolds

Firstly let's just clarify what an orbifold is. To any readers who read the end of the complex manifold notes, we have already seen that an orbifold is some sort of singular manifold, given by modding out by some finite group. It turns out that mathematicians and (string) physicists have slightly different definitions of an orbifold. To a mathematician the requirement is that this quotienting happens *locally*, whereas the physicists instead quotient the complete manifold globally. More explicitly:

> **Definition.** [Orbifold (Mathematician)] A (Hausdorff) topological space X is called an *orbifold* if, for each U_i in the covering open sets $\{U_i\}$ we have an open subset $V_i \subset \mathbb{R}^n$ and finite group Γ_i such that
>
>   (i) V_i is invariant under faithful, linear action of Γ_i,
>
>   (ii) there exists a homeomorphism $\varphi_i : V_i/\Gamma_i \to U_i$, known as the *orbifold chart*.

Basically all this definition is saying is that the neighbourhoods of $p \in U_i$ are given by $\mathbb{R}^n/\Gamma_i$, rather than simply $\mathbb{R}^n$ as for the manifold case.

> **Definition.** [Orbifold (String Theorist)] Let $\mathcal{M}$ be a manifold and let G be a group of (some of) the isometrics of $\mathcal{M}$. Then the orbit space $\mathcal{M}/G$ is called an *orbifold*.

In contrast to the mathematician's definition, we note quotient the whole manifold so that our orbifold is really a global notion. In either case, we get singularties arising in our orbifolds, given by the fixed points of the group action.

1.2 Sheaves

The language of sheaves will be used throughout these notes. I may come back later and add a section on sheaves, for completeness, but for time reasons now I simply refer any unfamiliar readers to Chapter 4 of my summer project on Twistor Theory

1.3 Algebraic Varieties

The above definition of an orbifold was given in terms of differential geometric language, i.e. in terms of manifolds. There is a closely related notion in *algebraic* geometry[1] called a subvariety, which we now outline how to define. To do this, we first introduce/recall a bunch of definitions (without further comment to save space).

Definition. [Algebraically Closed Field] Let K be a field, and denote the ring of polynomials in n-variables by $K[x_1, ..., x_n]$, i.e. $f \in K[x_1, ..., x_n]$ means $f(x_1, ..., x_n) \in K$. Then we call K *algebraically closed* if any non-constant polynomial over K has a root in K, i.e. $f(x_1, ..., x_n) = 0$ has $x_1, ..., x_n \in K$.

Example 1.3.1. The ring of real numbers is *not* algebraically closed as $f(x) = x^2 + 1$ has root $x = \pm i \notin \mathbb{R}$. However the ring of complex numbers is algebraically closed. In what follows we shall assume we are using the complex numbers everywhere.

Definition. [(Complex) Algebraic Variety] Consider the space $\mathbb{C}^n$,[2] and consider the algebraically closed field $\mathbb{C}$. Labelling the coordiantes of $\mathbb{C}^n$ by $\{z_1, ..., z_n\}$, we have $f \in \mathbb{C}[z_1, ..., z_n]$ being a $\mathbb{C}$ valued function over $\mathbb{C}^n$. Now consider some set of polynomials $S \subset \mathbb{C}[z_1, ..., z_n]$ and define for their common zero locus

$$Z(S) := \{z \in \mathbb{C}^n \,|\, f(z) = 0 \,\forall\, f \in S\}.$$

Then a subspace $X \in \mathbb{C}^n$ is called a *(complex) algebraic set* if $X = Z(S)$ for some S. If we can write U as the union of two proper algebraic sets then we say that U is *reducible*. If it is not reducible (and non-empty) it is irreducible. Finally, an irreducible algebraic set is called a *algebraic variety*. We can turn X into an topological space by defining our closed sets[3] to be the algebraic sets. This is known as the *Zariski topology*. A subspace of X that is also an algebraic variety is called an *algebraic subvariety*.

Terminology. From now one we shall simply say "variety" to mean "complex algebraic variety".

Now note that the dimension of the space X is related to the cardinality of the set $S \subset \mathbb{C}[z_1, ..., z_n]$. This is simply the statement that each zero condition allows us to relate

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[1] For those perhaps unfamiliar: as the name suggests differential geometry studies the geometry of differentials (i.e. vectors and tensors), whereas algebraic geometry is basically the study of the geometry associated to the solutions of polynomial equations. The two fields are obviously closely related and we often tread the line between the two. The algebraic equivalent to a manifold is called a *variety*.

[2] We define algebraic varieties more generally in terms of *affine spaces*, which $\mathbb{C}^n$ is an example of.

[3] For clarity, it is possible to define a topology in terms of closed sets, we simply reverse the conditions on unions and intersections.

one of the z_i to some of the others, and so reduces the dimension by one. We therefore see that a codimesion 1 subvariety is simply a hypersurface in X given by the zero locus of some polynomial. We saw ideas of this in the complex geometry notes when constructing Calabi-Yau manifolds as hypersurfaces of $\mathbb{CP}^n$. Indeed it turns out that for complex manifolds all hypersurfaces arise in this way, which we say again the following definition.

Definition. [Hypersurface] Given a complex space X, a *hypersurface*, Y, is a (sub)variety of codimension 1, i.e. $Y \subset X$ and $\dim Y = \dim X - 1$. A hypersurface is said to be *irreducible* if it corresponds to an irreducible (sub)variety.

A general hypersurface is given by the union of its irreducible components, i.e. $Y = \cup Y_i$ where Y_i are the irreducible hypersurfaces. If X is compact then any hypersurface has only finitely many irreducible components.

2 | Divisors

We are now in a position to start discussing divisors. We start with the definition of a Weil divisor.

2.1 Weil Divisors

> **Definition.** [(Weil) Divisor] A *(Weil) divisor*, D, on X is a formal linear combination of irreduicible hypersurfaces, i.e.
>
> $$D = \sum_i a_i[Y_i], \qquad \text{where} \qquad a_i \in \mathbb{Z}. \tag{2.1}$$
>
> The coefficients a_i give the order to which the defining polynomial vanishes, with negative values corresponding to poles. It is hopefully clear that we can turn this into a group in the natural way (i.e. by our addition), in this way we define the *divisor group* of X, $\mathrm{Div}(X)$.

We should clarify a bit the formal addition defined for Equation (2.1). This is simply defined in terms of the weightings of the defining polynomials and does not somehow correspond to "adding" two hypersurfaces together to get a new hypersurface. We can constrast this to the addition in homology, by recalling that hypersurfaces can be thought of in terms of homology (i.e. we can triangulate the hypersurface using simplicies). Homology has a well defined addition given in the expected way, namely $[Y_1]_{\mathrm{homol}} + [Y_2]_{\mathrm{homol}} = [Y_1 + Y_2]_{\mathrm{homol}}$. Now, let's imagine that Y_1 and Y_2 are two different hypersurfaces, but suppose that the corresponding elements in homology are in the same class, i.e. $[Y_1]_{\mathrm{homol}} = [Y_2]_{\mathrm{homol}}$. If we then consider their difference then we get a vanishing result in homology, but the *divisor* $D = a_1[Y_1] - a_2[Y_2]$ is non-vanishing. In this sense a divisor is a finer notion than a homology class. In this way we see that the square bracket notation in Equation (2.1) does *not* mean the corresponding homology class, however it is standard notation and so we keep it.

> **Definition.** [Support Of A Divisor] Let $D \in \mathrm{Div}(X)$ be a divisor, then the *support* of D is the subset of X given by
>
> $$\sup\left(\sum_i a_i[Y_i]\right) = \bigcup_{a_i \neq 0} Y_i.$$

Definition. [Effective Divisor] A divisor $D \in \text{Div}(X)$ is called *effective* is all $a_i \geq 0$.

If a divisor is effective, we write $D \geq 0$. More generally, given $D, D' \in \text{Div}(X)$ we write $D \geq D'$ if the difference $D - D'$ is effective.

Remark 2.1.1. Note given any hypersurface $Y = \cup_i Y_i$, we can define an effective divisor by $D = \sum [Y_i]$.

As the definition makes clear, a divisor is defined for *any* polynomial defined on our space, however not all polynomials correspond to *functions* on a space. That is, they don't correspond to sections of the constant sheaf $\mathcal{O}_X := \underline{\mathbb{C}}$. As an example, if our ambient space (i.e. the space we start with) is $\mathbb{CP}^n$, we can define a hypersurface, and therefore a divisor, by $z_0 = 0$, however this equation is not projectively well defined. We now want to describe divisors that arise from functions on our space.

Remark 2.1.2. To those that have read Proposition 3.2.3 of my complex manifolds notes (or to anyone who has seen the same result elsewhere), might want to say "but we've seen that polynomials *do* correspond to holomorphic sections!" The point is that what this proposition says is that a polynomial of degree d in $\mathbb{CP}^n$ is given by a holomorphic section of $\mathcal{O}_{\mathbb{CP}^n}(d)$, i.e. the tensor product of d copies of the hyperplane line bundle. What we're saying above is that this is not a holomorphic *function* on $\mathbb{CP}^n$, i.e. it is not a section of constant sheaf $\underline{\mathbb{C}}$. In fact we will actually *derive* the hyperplane line bundle result shortly.

2.1.1 Principal Divisors

Recall that a holomorphic function on $U \subseteq X$ is simply a $(0,0)$-form, i.e. a holormorphic section in $\Gamma(\Lambda^{0,0}, U)$. Perhaps more simply, a holomorphic function on some open set U is just a function who's charted image is holomorphic in $\mathbb{C}^{\dim X}$.

Definition. [Meromorphic Function] Let $U \subset \mathbb{C}^n$ be open, then a *mermorphic function* f on U is a function defined on the complement of a nowhere dense subset $S \subset U$ with the following property: there exists an open cover $U = \cup_i U_i$ and holormphic functions $g_i, h_i : U_i \to \mathbb{C}$ such that

$$h_i|_{U_i \backslash S} \cdot f|_{U_i \backslash S} = g_i|_{U_i \backslash S}.$$

We denote the set of all meromorphic functions on U by $K(U)$ and the sheaf of meromorphic functions by $\mathcal{K}(U)$.

Perhaps more intuitively, a meromorphic function is a holomorphic function that can be singular. Indeed for any point $z \in U$ we can write $f = g/h$ where g, h are holomorphic functions in U. In this way we can define the order of a meromorphic function as the order (i.e. "powers of z") of the quotient g/h. More preciely we have the following definitions. Here $Z(g) := \{z \in U \,|\, g(z) = 0\}$.

Definition. [Zero Set & Pole Set] Let $f \in Z(U)$ be given by $f = g/h$. Then the *zero set* $Z(f) \subset U$ is defined by $Z(g)$ while the *pole set* $P(f)$ is defined by $Z(h)$.

Definition. [Order Of Meromorphic Function] Let $f \in Z(U)$ be given by $f = g/h$. Then we define the *order of f on U* by $\mathrm{ord}_U(f) = \mathrm{ord}_U(g) - \mathrm{ord}_U(h)$.[1] We say that the meromorphic function has zeros/poles of order $\pm\mathrm{ord}_U(f)$ (where $\mathrm{ord}_U(f) \geq 0$).

If we now consider the specific cases when we are looking at a hypersurface in X, we can define $\mathrm{ord}_Y : K(X) \to \mathbb{Z}$, and so define a divisor as follows.

Definition. [Principal Divisor] Let $f \in K(X)$. then the divisor associated to f is

$$(f) := \sum \mathrm{ord}_Y(f)[Y], \tag{2.2}$$

where the sum is done over all irreducible hypersurfaces of X. We call a divisor of this form *principal*. We denote the group of principal divisors $\mathrm{Div}_0(X)$.

All we have said is that a principal divisor is simply the Weil divisor associated to a memromorphic function.

Now, note that a principal divisor (f) is not necessarily effective, but we can split it into the difference of two effective divisors, namely $(f) = Z(f) - P(f)$

$$Z(f) = \sum_{\mathrm{ord}_Y(f)>0} \mathrm{ord}_Y(f)[Y] \qquad \text{and} \qquad P(f) = \sum_{\mathrm{ord}_Y(f)<0} -\mathrm{ord}_Y(f)[Y]$$

known as the *zero divisor* and *pole divisor*, respectively. Next note that it follows from

$$\mathrm{ord}_Y(f_1 \cdot f_2) = \mathrm{ord}_Y(f_1) + \mathrm{ord}_Y(f_2)$$

for $f_1, f_2 \in K(X)$, that $(f_1 \cdot f_2) = (f_1) + (f_2)$. Putting this together with the fact that clearly the zero divisor $D = 0$ is the principal divisor associated to a constant function, we see that principal divisors form a subgroup of $\mathrm{Div}(X)$. We often denote this subgroup by $\mathrm{Div}_0(X)$.

Definition. [Linearly Equivalent Divisor] Let $D, D' \in \mathrm{Div}(X)$. We call them *linearly equivalent*, denoted $D \sim D'$, if $D - D'$ is a principal divisor.

Remark 2.1.3. Note that the name/notation is not a poor choice, indeed linear equivalence on divisors is an equivalence relation, i.e. clearly: $D \sim D$; $D \sim D' \iff D' \sim D$; and $D \sim D'$, and $D' \sim D''$ implies $D \sim D''$.

Definition. [Weil Divisor Class Group] We call the quotient of the Weil divisors by principal divisors the *Weil divisor class group*:

$$Cl(X) := \frac{\mathrm{Div}(X)}{\mathrm{Div}_0(X)}.$$

That is, an element in $Cl(X)$ is given by a class of linearly equivalent divisors.

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[1]Here the order of a holomorphic function can be thought of as the highest power in the series expansion.

2.2 Cartier Divisors

Now note that our construction of a principal divisor is a global one, namely we require that our meromorphic functions be global. We now want to try generalise this to the case when our meromorphic functions are only locally given. That is, we want to consider the case when we aren't given a global section in $\mathcal{O}_X$, but instead we only have collection of local sections in $\mathcal{O}_X(U_i)$ where $\cup U_i = X$. The idea is to take these local sections and "patch them together" to get a global one, thereby producing our global divisor. Well what we have just described is *exactly* the content of the zeroth Čech cohomology group, which motives the next proposition.

Proposition 2.2.1. *The following isomorphism exits*

$$H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*) \cong Div(X), \tag{2.3}$$

*we call the left-hand side the space of **Cartier divisors**.*

Proof. First we note that the space of non-trivial holomorphic functions are indeed a subset of the non-trivial meromorphic functions, i.e. (in terms of sheaves) $\mathcal{O}_X^* \subset \mathcal{K}_X^*$. This follows simply because we can think of a non-trivial holomorphic function as an invertible meromorphic function, and so the quotient $\mathcal{K}_X^*/\mathcal{O}_X^*$ on the left-hand side is well defined. For the rest of this proof we shall drop "non-trivial".

Now consider some element $f \in H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$. This is given by a collection of meromorphic functions $f_i \in \mathcal{K}_X^*(U_i)$ modulo multiplication by a holomorphic function on U_i, where $\{U_i\}$ is an open cover of X. By definition of Čech cohomology, f_i and f_j must agree on the overlap $U_i \cap U_j$, up to multiplication of an element in $\mathcal{O}_X^*(U_i \cap U_j)$, i.e. $f_i \cdot f_j^{-1}$ is a holomorphic function. This is a transition function and so, by definition, must be non-vanishing on the overlap. The fact that $f_i \cdot f_j^{-1}$ has no zeros means that $\mathrm{ord}_Y(f_i \cdot f_j^{-1}) = 0$ where Y is an irreducible hypersurface with $Y \cap U_i \cap U_j \neq 0$. However this tells us that $\mathrm{ord}_Y(f_i) - \mathrm{ord}(f_j) = 0$, which in turn tells us that the order of f itself is well defined, i.e. $\mathrm{ord}_Y(f_i) = \mathrm{ord}_Y(f_j)$. We can therefore define an element in $\mathrm{Div}(X)$ simply as $(f) = \sum \mathrm{ord}_Y(f)[Y]$. We have already seen that this is a group homomorphism, which follows from the fact that multiplication of meromorphic functions corresponds to addition in $\mathrm{Div}(X)$.

At first sight it might seem like this is a map onto $\mathrm{Div}_0(X)$ rather than the whole of $\mathrm{Div}(X)$, as required by Equation (2.3). This is not the case, and it follows from the fact that we are working in local patches rather than globally. We shall make this more clear now by defining the inverse map and showing it is a bijection.

Consider a $D = \sum a_i[Y] \in \mathrm{Div}(X)$. Then consider any irreducible hypersurface Y_i. This is defined by the vanishing of some polynomial on X. Now *locally* we can express this as the zero locus of a holomorphic function. As a concrete example, if we are considering $X = \mathbb{CP}^n$, then if we consider the patch $z_i \neq 0$, we can divide our polynomial through by this z_i to give us a projectively well defined result. This gives us a *locally* defined holomorphic function.[2] So if we consider some open cover $\{U_j\}$ for X, then the intersection $Y_i \cap U_j$ is defined by $g_{ij} \in \mathcal{O}(U_j)$ via

$$Y_i \cap U_j = g_{ij}^{-1}(0).$$

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[2] If this was a little hard to see, see the proof of proposition 3.2.3. of the Complex Manifolds notes.

This construction is unique, up to the multiplication of elements in $\mathcal{O}^*(U_j)$. Now define $f_j := \prod_i g_{ij}^{a_i} \in \mathcal{K}_X^*(U_j)$, which gives us a locally defined meromorphic function. Finally consider the intersection $U_j \cap U_k$: on this intersection g_{ij} and g_{jk} define the same irreducible hypersurface $Y_i \cap U_j \cap U_k$, and so must only differ by the multiplication of some element in $\mathcal{O}^*(U_j \cap U_k)$. In this way we can "glue together" the different f_js to a global element $f \in H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$.

Now clearly $\text{ord}_{Y_i}(f_j) = a_i$, and so its hopefully clear that if we apply the first map (i.e. from $H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$ to $\text{Div}(X)$) that we will get exactly $D = \sum a_i[Y]$ back, and so these two maps are inverses of each other, and seeing as both are injective we have our bijection. $\qquad\square$

2.2.1 Divisor Line Bundles

There is an alternative, sheaf theoretic, definition of Cartier divisors which will prove particularly useful for us. It involves introducing a couple definitions.

> **Definition.** [Picard Group] For a ringed space X, the *Picard group* is the group of isomorphism classes of invertible sheaves (i.e. line bundles) on X. Group multiplication is the tensor product. In other words,
>
> $$\text{Pic}(X) := H^1(X, \mathcal{O}_X^*). \tag{2.4}$$

Remark 2.2.2. The above two definitions are indeed compatible, let's see how. Let's consider some line bundle (i.e. an invertible sheaf) L and denote our local trivialisations by f_i, i.e. $f_i : U_i \to \mathcal{O}(U_i)$. Our transition functions are then given by $\psi_{ij} = f_i \cdot f_j^{-1}$ which is an automorphism on $\mathcal{O}(U_i \cap U_j)$, but this is just an element of $H^0(U_i \cap U_j, \mathcal{O}_X^*)$. That is, two holomorphic functions are related by a non-trivial holomorphic function. Finally we note that $\psi_{ik} = \psi_{ij}\psi_{jk}$, and so our cocycle condition is met and so we get an element in $H^1(X, \mathcal{O}_X)$.

> **Definition.** [Fractional Ideal Sheaf] Let $\mathcal{O}_X$ and $\mathcal{K}_X$ denote the sheaf of holomorphic and meromorphic functions on X. Then a *fractional ideal sheaf* is a sub-$\mathcal{O}_X$-module[3] of $\mathcal{K}_X$. A fractional ideal sheaf, $\mathcal{F}$, is *invertible* if for all $x \in X$ there is an open neighbourhood $x \in U$ such that the restriction of $\mathcal{F}$ to U is given by $\mathcal{O}_X(U) \cdot f$ where $f \in \mathcal{K}_X^*$.

We now note that the definition of an invertible fractional ideal sheaf is essentially equivalent to the definition of a Cartier divisor. That is, thinking of a Cartier divisor as the collection $\{U_X(U_i), f_i\}$ we can define an invertible fractional ideal sheaf piece-wise. Similarly, given a invertible fractional ideal sheaf, we can define a Cartier divisor. The important thing for us is that, in this way, to a given divisor $D \in \text{Div}(X)$ we can define a line bundle $\mathcal{O}(D) \in \text{Pic}(X)$.[4] The following Lemma makes this more explicit.

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[3]Let R be a ring, then an R-module is some abelian group $(A, +)$ equipped with an operation $\cdot : R \times A \to A$ that satisfies the usual stuff (distributive etc). It is hopefully clear that $\mathcal{K}_X$ is a $\mathcal{O}_X$-module, and so we can define a sub-$\mathcal{O}_X$-module.

[4]We may also sometimes use the notation $\mathcal{L}(D)$ to denote the line bundle associated to a divisor.

> **Lemma 2.2.3.** *There is a group homomorphism given by*
>
> $$Div(X) \to Pic(X)$$
> $$D \mapsto \mathcal{O}(D),$$
>
> *where $\mathcal{O}(D)$ is the invertible fractional ideal sheaf associated to D.*

Proof. First we use that a $D \in \mathrm{Div}(X)$ can be thought of as a Cartier divisor, i.e. a $f \in H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$. As we explained in the proof of Equation (2.3), such an f is given by a collection of $f_i \in \mathcal{K}^*(U_i)$ modulo multiplication by an element of $\mathcal{O}_X^*$. In other words, f restricted to U_i is given by $\mathcal{K}^*(U_i) \cdot g$ where $g \in \mathcal{O}_X^* \subset \mathcal{K}_X^*$, but this is exactly our invertible ideal sheaf condition, and so we define a map $D \mapsto \mathcal{O}(D)$. This is an element of the Picard group as it is described exactly by the transition functions $\psi_{ij} = f_i \cdot f_j^{-1}$ introduced above.

We now just need to show it is a group homomorphism, i.e. it respects the group structures. That is, we need to show that

$$\mathcal{O}(D + D') = \mathcal{O}(D) \otimes \mathcal{O}(D'). \tag{2.5}$$

Well, let's assume the Cartier divisors D and D' are given by a common open covering[5] and transition maps $\{f_i\}$ and $\{f_i'\}$. Now recalling (see proof of Equation (2.3)) that we define $f_i := \prod_j g_{ji}^{a_j}$, it follows that the transition functions of $D + D'$ are given by $\{f_i \cdot f_i'\}$, but then $\mathcal{O}(D + D')$ corresponds to $\{(f_i \cdot f_i') \cdot (f_j \cdot f_j')^{-1}\} = \{\psi_{ij} \cdot \psi_{ij}'\}$. This latter term itself describes $\mathcal{O}(D) \otimes \mathcal{O}(D')$, and so we're done. $\square$

Remark 2.2.4. Note it is important not to confuse the subscripts i, j on ψ/ψ' with components of a matrix. In the sense that we defined the transition function for the tensor product of two bundles by a matrix who's entries are given by $(g_1)_{ij}\mathbf{g_2}$. Here our transition functions are sections of $\mathcal{O}_X^*(U_i \cap U_j)$ and so only have one component. The i, j labels are labelling the open sets U_i, U_j. This is why $\{\psi_{ij} \cdot \psi_{ij}'\}$ does indeed correspond to $\mathcal{O}(D) \otimes \mathcal{O}(D')$.

Note that $\mathcal{O}(D - D) = \mathcal{O}(0) \cong \mathcal{O}_X$, and so it follows from Equation (2.5) that $\mathcal{O}(-D) = \mathcal{O}(D)^*$ where $*$ denotes dual. This should start to ring some resemblance to the discussion of tautological and hyperplane line bundles in the complex manifolds notes: Lemma 2.2.3 basically tells us that we can associate a line bundle to a divisor. This is exactly what we do for the hyperplane line bundle, it is exactly the bundle corresponding to a hyperplane divisor (hence the name). Let's outline this a bit more clearly now.

Example 2.2.5. Consider $\mathbb{CP}^n$ and consider the hyperplane given by $H = \{z_0 = 0\}$. This corresponds to the divisor $1 \cdot H = H$. Our charts are given by $U_i = (z)/z_i$, where $(z) = (z_0, ..., z_n)$, so our $f_i = z_0/z_i$, and our transition functions are given by $\psi_{ij} = (z_0/z_i)(z_0/z_j)^{-1} = z_j/z_i$, but these are exactly the transition functions for the hyperplane line bundle $\mathcal{O}(1)$. Really when we write $\mathcal{O}(1)$ we mean $\mathcal{O}(1 \cdot H)$, and similarly $\mathcal{O}(n)$ means $\mathcal{O}(n \cdot H)$. Then the comment above then confirms that $\mathcal{O}(-1) = \mathcal{O}(-1 \cdot H)$ is the dual of $\mathcal{O}(1)$.

---

[5]This can be done by passing to a refinement.

We can, in fact, show Lemma 2.2.3 in a nice way by considering the following short exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_X^* \longrightarrow \mathcal{K}_X^* \longrightarrow \mathcal{K}_X^*/\mathcal{O}_X^* \longrightarrow 0,$$

where exactness follows from $\mathcal{O}_X^* \to \mathcal{K}_X^*$ being injective and $\mathcal{K}_X^* \to \mathcal{K}_X^*/\mathcal{O}_X^*$ being surjective. This short exact sequence in sheaves gives rise to a long exact sequence in cohomology, the first few elements of which are

$$0 \longrightarrow H^0(X,\mathcal{O}_X^*) \longrightarrow H^0(X,\mathcal{K}_X^*) \overset{\varphi}{\longrightarrow} H^0(X,\mathcal{K}_X^*/\mathcal{O}_X^*) \overset{\delta}{\longrightarrow} H^1(X,\mathcal{O}_X^*) = \mathrm{Pic}(X) \longrightarrow ...,$$

and so Lemma 2.2.3 is seen simply as the map $\delta : H^0(X,\mathcal{K}_X^*/\mathcal{O}_X^*) \to H^1(X,\mathcal{O}_X^*)$. We can also see from here that a principal Cartier divisor (i.e. one that corresponds to an element of $\mathrm{Div}_0(X)$) is a Cartier divisor that lies in the image of the map $\varphi : H^0(X,\mathcal{K}_X^*) \to H^0(X,\mathcal{K}_X^*/\mathcal{O}_X^*)$. This leads nicely into the next Lemma.

Lemma 2.2.6. *A divisor $D \in Div(X)$ is principal if and only if $\mathcal{O}(D) \cong \mathcal{O}_X$.*

Proof. Firstly we note that $\mathcal{O}(D) \cong \mathcal{O}_X$ is the identity element of the Picard group. This follows simply from the definition of an invertible fractional ideal sheaf: if $\mathcal{F} \cong \mathcal{O}_X$ then clearly $\mathcal{F}$ restricted to U is isomorphic to $\mathcal{O}_X(U)$, i.e. we take the identity element in $\mathcal{K}_X^*$.

Now we just use our exact sequence above. If D is principal then it lies in the image of $\varphi : H^0(X,\mathcal{K}_X^*) \to H^0(X,\mathcal{K}_X^*/\mathcal{O}_X^*)$. However it then follows from exactness that this divisor is in the kernel of $\delta : H^0(X,\mathcal{K}_X^*/\mathcal{O}_X^*) \to \mathrm{Pic}(X)$, and so we have $\mathcal{O}(D) \cong \mathcal{O}_X$ if D is principal.

Now assume that $\mathcal{O}(D) \cong \mathcal{O}_X$. Then the $D \in \mathrm{Div}(X)$ is given by a $f \in H^0(X,\mathcal{K}_X^*/\mathcal{O}_X^*)$ with transition functions $\psi_{ij} = g_i \cdot g_j^{-1}$, where the $g_i \in \mathcal{O}_X^*(U_i)$ are the unit elements, i.e. g_i^{-1} just restricts us to U_i. Putting this together with the fact that $\psi_{ij} = f_i \circ f_j^{-1}$ we can equate the two and obtain $f_i \cdot g_i^{-1} = f_j \cdot g_j^{-1}$ on the intersection, but this is just the statement that $f_i = f_j$ on the intersection and so our f is just a globally defined meromorphic function, and therefore $D = (f)$, i.e. it is principal. $\square$

We then have the following, useful, corollary.

Corollary 2.2.7. *The group homomorphism from Lemma 2.2.3 provides an injection*

$$\phi : Cl(X) \to Pic(X), \tag{2.6}$$

where $Cl(X)$ is the Weil divisor class, $Cl(X) = Div(X)/Div_0(X)$.

We note that this injection is, in general, a strict inclusion. However we now have the following proposition.

Proposition 2.2.8. *If a line bundle $\mathcal{L}$ admits a global section it is contained in the image of Equation (2.6).*

Proof. The key thing to note is that a global section in a line bundle is a hyperplane of the line bundle, and so corresponds to some divisor. Let s be such a non-zero global section and denote by D_s the associated divisor. This establishes a link between the section of a divisor, what we want is a link between the *line bundle* and D_s. Well, any line bundle can be defined

by its sections, and any two sections are related by by a meromorphic function, which is itself an element in $\text{Pic}(X)$ (it's a section in $\mathcal{K}(X)$). So if we consider a $\tilde{s} = f \otimes s$, then we have, recalling the group structure on each space,

$$\phi([D_s]) = \phi([D_f + D_s]) = f \otimes s = \tilde{s} = \phi([D_{\tilde{s}}]),$$

where we have used that $D_f \sim 0$ as viewed as an element in $Cl(X)$. Then using that our map is injective, we have $D_s \sim D_{\tilde{s}}$, and so we really are talking about the whole line bundle. $\square$

2.2.2 Divisor Chern Class

The last thing we want to discuss before moving on to toric geometry is the Chern class of a divisor. We note that this is a well defined thing, as we have just shown that a divisor D has a corresponding line bundle $\mathcal{O}(D)$, and we know how to define Chern classes for line bundles.

Ok so what is Chern class of a divisor? Well line bundles have total Chern class $c(\mathcal{L}) = 1 + c_1(\mathcal{L})$, as they are 1-dimensional. Then we recall that the first Chern class is a $(1,1)$-form, which is Poincaré dual real codimension-2 space, which is a complex hyperplane. In other words, the first Chern class is a divisor. Putting this together with the fact that the Chern classes are directly related to the connection, which in turn defines our horizontal subspace, which are related to our sections, it is not a great leap to make the following claim.

> <u>*Claim 2.2.9*</u>. Let $\text{Pic}^*(X)$ denote the group of isomorphism classes of line bundles that admit a global section. Then the group homomorphism
>
> $$c_1 : \text{Pic}^*(X) \to Cl(X)$$
>
> is the first Chern class of the line bundle.

<u>*Remark 2.2.10*</u>. Recall that the Picard group is given by the cohomology class $H^1(X, \mathcal{O}_X^*)$. In this manner, the first Chern class can also been seen as a map between cohomologies. We have the short exact sequence of sheaves

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0,$$

which induces a long exact sequence in cohomology, containing a morphism

$$c_1 : H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z}).$$

This is the first Chern class. The divisor class group corresponds to real codimensional-2 hypersurfaces. These are Poincaré dual to 2-forms, and so this map makes sense.

This is a very useful result. In particular, note that we can express the total Chern class for a CY surface inside a projective space in terms of divisors. Explicitly, let A be an n-complex-dimensional projective space defined as the direct sum of m projective spaces B_j. We can think of A as being the union of $(n+1)$ divisors D_i, one for each hypersurface $z_i = 0$, where z_i are the homogeneous coordinates of A. This is equivalent to saying that we can

think of it as the direct sum of $(n+1)$ line bundles, one for each homogeneous coordinate. Then recalling that $c(L_1 \oplus ... \oplus L_{n+1}) = \prod_{i=1}^{n+1}(1 + c_1(L_i))$, we have

$$c(A) = \prod_{i=0}^{n}(1 + D_i). \tag{2.7}$$

Similarly, the hypersurface is given by a polynomial P of set degree p_j in each B_j factor. Each of these define a Weil divisor H_j of order p_j. Note that these H_js are divisors in the components B_js, while the polynomial corresponds to a divisor in the full A, and so we have

$$c(P) = 1 + \sum_{j=1}^{m} p_j H_j. \tag{2.8}$$

These divisors H_j are clearly related to the divisors D_i, and we will see exactly how later.

Let's give a couple familiar examples to help ground this.

Example 2.2.11. Let's consider the CY defined by the quintic in $A = \mathbb{CP}^4$. The hyperplanes $z_i = 0$ are all linearly equivalent (simply by redefinition of the coordinates) and so $D_0 = D_1 = D_2 = D_3 = D_4 = D$, and so

$$c(\mathbb{CP}^4) = (1 + D)^5$$

We then have the associated line bundle $\mathcal{O}(D) \equiv \mathcal{O}(1)$, where the equivalence was explained in Example 2.2.5. This then returns the familiar result $c(\mathbb{CP}^4) = (1 + \mathcal{O}(1))^5$.

Then the hypersurface is simply given by a degree 5 polynomial in $\mathbb{CP}^4$, and clearly $H_j = D$, as this is the only divisor available to us. So we have

$$c(P) = 1 + 5D,$$

which corresponds to the familiar result $c(P) = 1 + 5\mathcal{O}(1)$.

Remark 2.2.12. Note we are being a bit clumsy with notation as $\mathcal{O}(D) \in \text{Pic}^*(X)$ while $D \in Cl(x)$ and so we should really have different notation for the c appearing with Ds and the c appearing with $\mathcal{O}(D)$s. However this would just add more notation, and it is hopefully clear what is meant, so we continue with this notation.

Example 2.2.13. Let's now consider a CY in $A = \mathbb{CP}^3 \oplus \mathbb{CP}^1$. Here we have two inequivalent D_i divisors: for for hypersurfaces in $\mathbb{CP}^3$ and one for hypersurfaces in $\mathbb{CP}^1$. Let's denote these D_3 and D_1, respectively. Then

$$c(A) = (1 + D_3)^4(1 + D_1)^2 \qquad \rightarrow \qquad \left(1 + \mathcal{O}_{\mathbb{CP}^3}(1)\right)^4\left(1 + \mathcal{O}_{\mathbb{CP}^1}(1)\right)^2,$$

which again is a familiar result. Our defining polynomial is then of degree $(4, 2)$, and we have $H_3 = D_3$ and $H_1 = D_1$, giving us

$$c(P) = 1 + 4D_3 + 2D_1 \qquad \rightarrow \qquad 1 + 4\mathcal{O}_{\mathbb{CP}^3}(1) + 2\mathcal{O}_{\mathbb{CP}^1}(1).$$

We now note an important point. We defined the first Chern class above for line bundles that admit a global section, however we did _not_ say that this global section need not vanish anywhere. In other words, we are not only considering trivial line bundles, as this would be a

bit boring. In fact, recall that a divisor is essentially defined by the order of zeros and poles. We call the sum of these orders the *degree* of the divisor. If we have a holomorphic section we have no poles, and so the degree is just given by the sum of the zeros. In other words, the order of a divisor is given by the number of zeros of the section. We then make the (not so hard to believe, see Chapter 1 of [1] for more details) claim that the integral of the first Chern class of a line bundle corresponding to a divisor gives you the degree of the divisor.

This might all seem like somewhat unnecessary new notation. However, we will see later when considering complicated hypersurfaces in toric varieties that the ability to express the Chern class in terms of divisors will greatly simplify things.

3 | Toric Varieties

We are now ready to start actually discussing toric geometry, in particular we want to construct toric varieties and look how they encode orbifolds. We then want to discuss how to resolve these orbifold singularities, and also discuss the construction of Calabi-Yau spaces from this perspective. As we will see, the tools of toric geometry make the construction of Calabi-Yau manifolds of desired properties an almost trivial combinatorics game, and so it is a very powerful tool.

There are two main approaches to the study of toric goemetry: the spectrum approach and the coordinate approach. The former deals with a lot more algebraic geometry directly, while the latter is probably more intuitive, especially for a first time approach to the subject. For that reason, we shall focus almost entirely on the latter approach.

3.1 Basic Definitions

The most important thing for us to clear up immediately is what we mean by a torus. We do *not* mean the topological torus $S^1 \times S^1$ that we are (hopefully) familiar with. Instead we are talking about the *algebraic* torus, which we now define.

> **Definition.** [Algebraic Torus] An *algebraic n-torus T* is given by the n-fold product of $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$. That is $T = (\mathbb{C}^*)^n$, and we regard this as an abelian group.

The obvious question to ask is "why on Earth do we call this a torus?" Well the idea is that one can construct the more familiar notion of a torus using the complex planes. For example, it is a fact that the 2-torus can be viewed $\mathbb{C}$ with equivalence relations

$$z \sim z + 2\pi \qquad \text{and} \qquad z \sim z + 2\pi\tau \qquad \text{with} \qquad \tau \in \mathbb{C} \setminus \mathbb{R}.$$

This is not as crazy as it might sound: if we take $\tau = i$ then we are simply taking the complex plane and defining two orthogonal S^1s, one that runs along the real axis and one along the imaginary axis. This gives us precisely the flat torus $S^1 \times S^1$.

If the above explanation doesn't make sense, it is not a problem, we will not really be concerned with it anymore. From now on whenever we say "torus" we mean "algebraic torus".

> **Definition.** [Toric Variety] Let X be a $\mathbb{C}$ variety. Then we call X a *toric variety* if it contains an n-torus T as a dense open subset, such that the natural action of the torus on itself (i.e. simply multiplication in $(\mathbb{C}^*)^n$) extends to an action of T on the whole of X.

Example 3.1.1. Perhaps the most important example of a toric variety for us will be $\mathbb{CP}^n$ and $\mathbb{WCP}^n$. We show here that the former is indeed a toric variety. This is actually very straight forward: let's denote the homogeneous coordinates of $\mathbb{CP}^n$ by $[z_0 : ... : z_n]$. Then note that the open subset

$$T = \{[z] \,|\, z_i \neq 0 \,\forall i\} = (\mathbb{C}^*)^{n+1}/\mathbb{C}^* \subset \mathbb{CP}^n$$

where the quotient $\mathbb{C}^*$ is embedded diagonally into $(\mathbb{C}^*)^{n+1}$, is dense and is clearly isomorphic to $(\mathbb{C}^*)^n$, and so is an algebraic torus. We can have this act on $\mathbb{CP}^n$ simply by coordinatewise multiplication, so we see that $\mathbb{CP}^n$ is a toric variety. We similarly have that $\mathbb{WCP}^n$ is a toric variety.

> **Definition.** [Cone] Let N be a rank r lattice, and define $N_\mathbb{R} := N \otimes \mathbb{R}$. Then a *(strongly convex, rational, polyhedral) cone* $\sigma \in N_\mathbb{R}$ is a set
>
> $$\sigma := \{a_1 v_1 + a_2 v_2 + ... + a_k v_k \,|\, a_i \geq 0 \,\forall i\} \qquad \text{such that} \qquad \sigma \cap (-\sigma) = \{0\},$$
>
> where $\{v_1, ..., v_k\} \subset N$ is a finite set of vectors called the *generators* of σ. We call the boundary of a cone a *face*, and similarly we call a 1D cone an *edge* or a *ray*.

> **Definition.** [Fan] A collection Σ of cones in $N_\mathbb{R}$ is called a *fan* if:
>
> (i) each face of a cone in Σ is also a cone in Σ; and
>
> (ii) the intersection of any two cones in Σ is a face in each of the cones.

Remark 3.1.2. Note that we consider a cone to be a face of itself. This sounds funny at first, but note it is exactly what we need if we want to be able to say that a face is a cone. In other words, consider the intersection of a cone σ with a proper face $F \neq \sigma$, $\sigma \cap F = F$. If we did not consider F to be a face of F then we would fail to satisfy (ii) above.

> **Definition.** [Simplical Cone/Fan] A cone σ is called *simplical* if it can be generated by a set of vectors $\{v_1, ..., v_k\}$ which form a basis for the vector space they span. A fan Σ is called simplical if all $\sigma \in \Sigma$ are simplical. In these notes we will basically only consider simplical fans.

Example 3.1.3. An example of a fan in 2D with 7 generating vectors is the following
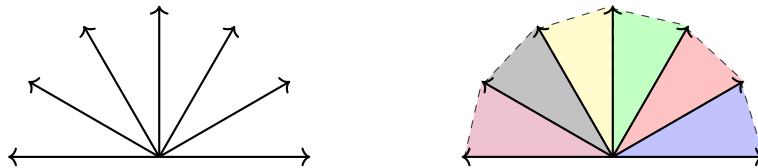
The diagram on the left just shows the generating vectors, while the right-hand diagram also shows the 2D cones, indicated by the shaded regions. This fan contains a total of 14 cones: the 6 triangular faces, the 7 edges and the origin. This example explains why we call it a fan: this looks like a Chinese fan!

Example 3.1.4. An example of a fan in 3D with 4 generating vectors is the following drawing: the left hand side are our 4 generating vectors, while the right is the full fan.

This fan contains 9 cones: the 4 triangular faces, the 4 edges, and the origin. It is hopefully clear from the diagram that these cones obey they required properties to be a fan, if not this is a bonus exercise to check.

Now we have defined our fan in terms of $N_\mathbb{R}$, which was in turn defined by a lattice N. The obvious question to ask is "can we define a cone directly from the lattice N?" The answer is "sometimes", which we now clarify.

For simplicity, consider a 2D lattice. Now take some lattice point, say $(1, 2)$. Now draw a line ρ from the origin through this lattice point and off to infinity.
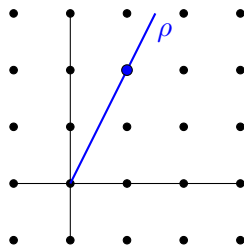
This line looks exactly like a 1D cone corresponding to the vector $v = (1, 2)$. Indeed ρ is exactly the ray corresponding to this vector. We call the lattice point the *ray generator*. If we then consider multiple ray generators we can build up our fan. So we see that we can make a fan by considering the lattice points, but it is hopefully clear that we can *not* always take a fan and associate a ray generator to each ray: generically if we have multiple ways, there is no way to put them on a lattice such that each ray passes through a lattice point. We call the cones/rays/fans that *can* be generated by a lattice *rational*. That is, we can identify a rational fan $\Sigma \subset \mathbb{R}^n$ by a set of points in $\mathbb{Z}^n$. Unless otherwise specified, we shall assume we are dealing with rational fans in these notes.

Finally it will be useful for the construction of toric varieties in the next section to introduce the dual lattice to N. By definition, this dual lattice is just given by $M := \hom(N, \mathbb{Z})$, where hom standards for "homomorphism". We shall denote the inner pairing of N and M by $\langle \, , \, \rangle : M \times N \to \mathbb{Z}$, and we also define the vector space $M_\mathbb{R} = M \otimes \mathbb{R}$.

3.2 Constructing Toric Varieties Using Fans

Ok great, so we've introduced toric varieties and then introduced our fans, the obvious question is "what do these have to do with each other?" At first sight there doesn't seem to be any clear relation between the two, but we shall now show that fans give us a very elegant

and simple way to construct toric varieties (and then Calabi-Yau hypersurfaces inside these toric varieties) with all kinds of desired properties.

Before doing so, we want to stress that it is this link between the borderline trivial combinatorics game of picking points on a lattice and toric varieties, that gives us immense power when constructing Calabi-Yau manifolds. This connection to algebraic geometry is missing in the closely related study of G2/Spin(7) manifolds, and, as such, the latter are much harder to construct explicitly and study. We can either see this as a good thing or a bad thing: it's bad because its more of a challenge, but it's good because it might suggest there is a deeper idea associated with G2/Spin(7) manifolds. We will not discuss these more here, but I do plan to write some notes on these when I understand them better.[1]

Ok so how do we make a toric variety from a fan? Again there are two ways to do this: using the spectrum construction or the homogeneous coordinate construction. We will be focusing on the latter.

Consider some fan Σ, and let $\Sigma(1)$ denote the set of edges in this fan. For concreteness, let $n = |\Sigma(1)|$. Now recall that every edge has an associated generating vector, so we have a set $\{v_1, ..., v_n\}$ that described $\Sigma(1)$. It is convenient to provide an ordering for these vectors, and we shall often do this implicitly. Now the key point is that to each edge ρ (or equivalently the associated vector) we associate a homogeneous coordinate z_ρ, so in total we have n coordinates $\{z_1, ..., z_n\}$ describing $\Sigma(1)$. This is clearly a start to defining a variety, but what do we do with these coordinates? Well, we next we introduce the following definition.

> **Definition.** [Exceptional Set] Let $\mathcal{S} \subseteq \Sigma(1)$ denote any subset that does *not* span a cone in Σ. That is $\{\rho_1, \rho_2\} \in \mathcal{S}$ if we do not have a 2D cone given by the face joining ρ_1 and ρ_2. Then we define $V(\mathcal{S}) \subset \mathbb{C}^n$ to be the linear subspace defined by setting $z_\rho = 0$ for all $\rho \in \mathcal{S}$. Finally we define the *exceptional set of* Σ to be
>
> $$Z(\Sigma) = \bigcup V(\mathcal{S}). \tag{3.1}$$

Remark 3.2.1. If the above definition is a little hard to follow on first reading, it will become clearer in the examples given below.

Our toric variety is then going to be defined as a quotient of the space $\mathbb{C}^n \setminus Z(\Sigma)$. We just have to define the quotienting group. The idea is to consider the map

$$\phi : \hom(\Sigma(1), \mathbb{C}^*) \to \hom(M, \mathbb{C}^*).$$

This is a map of maps, and is defined by mapping

$$\phi : \big(f : \Sigma(1) \to \mathbb{C}^*\big) \mapsto \left(m \mapsto \prod_{\rho \in \Sigma(1)} f(v_\rho)^{\langle m, v_\rho \rangle}\right).$$

---

[1]Obviously edit this sentence if/when done.

If we work in terms of coordinates, so that $v_j = (v_{j1}, ..., v_{jr})$ we can write ϕ explicitly as

$$
\begin{aligned}
\phi : (\mathbb{C}^*)^n &\to (\mathbb{C}^*)^r \\
(t_1, ..., t_n) &\mapsto \left(\prod_{j=1}^n t_j^{v_{j1}}, ..., \prod_{j=1}^n t_j^{v_{jr}} \right),
\end{aligned}
\tag{3.2}
$$

where the dimensions follow from the fact that $n = |\Sigma(1)|$ and N/M are rank r lattices, and the notation follows from the fact that $(\mathbb{C}^*)^k$ is a torus. We then define our quotienting group by

$$
G := \ker \left(\hom(\Sigma(1), \mathbb{C}^*) \xrightarrow{\phi} \hom(M, \mathbb{C}^*) \right).
\tag{3.3}
$$

Let's check that this does actually give a well defined action on $\mathbb{C}^n \setminus Z(\Sigma)$. It follows from the definition that $G \subset \hom(\Sigma(1), \mathbb{C}^*)$, and so given a $g \in G$ and a $\rho \in \Sigma(1)$ we can define $g(v_\rho) \in \mathbb{C}^*$. We then use this to define an action of G on $\mathbb{C}^n$ simply by

$$
g(z_1, ..., z_n) = \big(g(v_1)z_1, ..., g(v_n)z_n \big).
$$

Finally, recalling the definition Equation (3.1), it is clear that this action is closed in $\mathbb{C}^n \setminus Z(\Sigma)$; that is, $Z(\Sigma)$ just makes it so that certain elements can't vanish together, but $g(v_\rho) \neq 0$, and so this won't change this behaviour. We now finally arive at the definition of a toric variety associated to a fan.

> **Definition.** [Toric Variety From Fan] Let Σ be some fan with $n = |\Sigma(1)|$, and define $Z(\Sigma)$ and G via Equations (3.1) and (3.3), then
>
> $$
> X_\Sigma := \frac{\mathbb{C}^n \setminus Z(\Sigma)}{G}
> \tag{3.4}
> $$
>
> is a toric variety. The dense open torus is simply given by $T := (\mathbb{C}^*)^n/G \subset X_\Sigma$, and it acts on X_Σ by coordinatewise multiplication. It follows from Equation (3.2) that T has rank r ans X_Σ is an r-dimensional toric variety.

Remark 3.2.2. Note that the definition Equation (3.4) really is a property of Σ _not_ just the rays used to generate it. This enters into the fact that the exceptional set $Z(\Sigma)$ depends explicitly on all the cones in Σ, not just the edges.

Remark 3.2.3. We should clarify that the resulting toric variety X_Σ not only depends on the fan Σ but actually depends on the lattice N that Σ lies in. In particular, let $\widetilde{N} \subseteq N$ be a sublattice of finite index,[2] then we have $N_\mathbb{R} = \widetilde{N}_\mathbb{R}$, and so any fan $\Sigma \subseteq N_\mathbb{R}$ can also be considered a fan in $\widetilde{N}_\mathbb{R}$. The important point is that we associate certain properties to X_Σ depending on how Σ sits on the _lattice_ N or $\widetilde{N}$. The two resulting varieties $X_{\Sigma,N}$ and $X_{\Sigma,\widetilde{N}}$

---

[2]This essentially just means N is $\widetilde{N}$ with some points removed. The index of the sublattice is given by the number of cells in $\widetilde{N}$ that fit into one cell in N. For example if we have $N = \mathbb{Z}^2$ then $\widetilde{N} = \mathbb{Z}(1,2) = \{(n, 2n) \,|\, n \in \mathbb{Z}\}$ is a sublattice of index 2.

are, of course, related, and we will see exactly how later. For now we shall just assume we are working with the "nicest" lattice, i.e. the smallest lattice such that all generating vectors lie on lattice points.

Before going on to consider some examples, we include a quick proposition that will be useful for us later.

Proposition 3.2.4. *Let N and $\widetilde{N}$ be two lattices, which need not have the same dimension. Then suppose we have two fans $\Sigma \subseteq N_{\mathbb{R}}$ and $\widetilde{\Sigma} \subseteq \widetilde{N}_{\mathbb{R}}$, then*[3]

$$\Sigma \times \widetilde{\Sigma} := \{\sigma \times \widetilde{\sigma} \,|\, \forall\, \sigma \in \Sigma \ and \ \widetilde{\sigma} \in \widetilde{\Sigma}\} \subseteq N_{\mathbb{R}} \times \widetilde{N}_{\mathbb{R}}$$

if a fan, and we have

$$X_{\Sigma \times \widetilde{\Sigma}} = X_{\Sigma} \times X_{\widetilde{\Sigma}}.$$

3.2.1 Some Examples

Ok that was quite a bit of information, so let's give some examples to help ground all this information.

Example 3.2.5. Let's start with something we already know is a toric variety $\mathbb{CP}^n$. For ease of drawing, we consider $\mathbb{CP}^2$. This corresponds to a 2-dimensional lattice with 3 generating vectors. We claim that this corresponds to the following fan
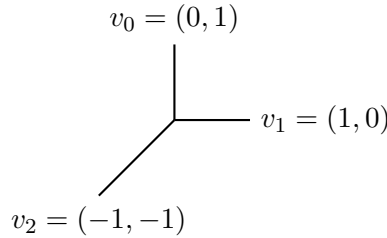
where we have only drawn the edges, the rest of the cone is given by the 3 faces given by pairing two of the edges.[4] Let's now verify that this is indeed $\mathbb{CP}^2$.

Firstly we note that the exceptional set is just point $\{0, 0, 0\}$, as the only combination of edges which doesn't span a cone is $\mathcal{S} = \{(0, 1), (1, 0), (-1, -1)\}$. For clarity, this is *not* a cone in Σ for two reasons: firstly if we defined $\sigma_{012} = av_0 + bv_1 + cv_2$ then we would fail to satisfy $\sigma_{012} \cap (-\sigma_{012}) = \{(0, 0)\}$; secondly even if it was a cone the intersection of σ_{012} and σ_{01}[5] is σ_{01}, but this is clearly not a face in σ_{01}. So we have $\mathbb{C}^3 \setminus Z(\Sigma_{\mathbb{CP}^2}) = \mathbb{C}^3 \setminus \{(0, 0, 0)\}$.

Next we need to find the group G. From Equation (3.2) we have

$$\phi : (\mathbb{C}^*)^3 \to (\mathbb{C}^*)^2$$
$$(t_0, t_1, t_2) \mapsto (t_2^{-1} t_1, t_2^{-1} t_0),$$

---

[3]The product $\sigma \times \widetilde{\sigma}$ is meant in the obvious way, i.e. we consider place the generating vectors of σ and $\widetilde{\sigma}$ perpendicular to each other and consider all cones from these generators.

[4]Bonus exercise: check that this is indeed a fan.

[5]Hopefully this notation is clear, but as we will use it a lot going forward, we explain it once: this simply means the cone with generating vectors v_0 and v_1.

which follows from $v_0 = (v_{01}, v_{02}) = (0, 1)$ etc. Then G is defined as the kernel of this map, i.e. we want the right-hand element to be $(1, 1)$, which clearly requires $t_0 = t_1 = t_2$, in other words $G = \{(t, t, t) \,|\, t \in \mathbb{C}^*\}$ which clearly isomorphic to $\mathbb{C}^*$, and so $G \cong \mathbb{C}^*$. We therefore arrive as

$$X_{\Sigma_{\mathbb{CP}^2}} = \frac{\mathbb{C}^3 \setminus \{(0, 0, 0)\}}{\mathbb{C}^*} = \mathbb{CP}^2.$$

Finally note that $T = (\mathbb{C}^*)^3/G = (\mathbb{C}^*)^3/\mathbb{C}^*$, where the $\mathbb{C}^*$ is embedded diagonally into $\mathbb{C}^*$, which is exactly what we had in Example 3.1.1.

Exercise

Verify that

$$v_0 = (-1) \quad\overline{\rule{0pt}{1em}\hspace{6em}\bullet\hspace{6em}}\quad v_1 = (1)$$

is the fan corresponding to $\mathbb{CP}^1$, and check the corresponding torus agrees with Example 3.1.1.

Example 3.2.6. Next let's look at a weighted projective space. Again for ease of drawing we consider $\mathbb{WCP}^2_{3,2,1}$. This again corresponds to a 2D lattice with 3 generating vectors. If we look through the details of Example 3.2.5, we see that the weightings of the coordinates enters in by the mapping ϕ, which is directly related to the entries of the vectors $\{v_0, v_1, v_2\}$. We therefore just want to make it so that v_2 is three times v_0 in one entry and twice v_1 in the other. That is, we consider the diagram
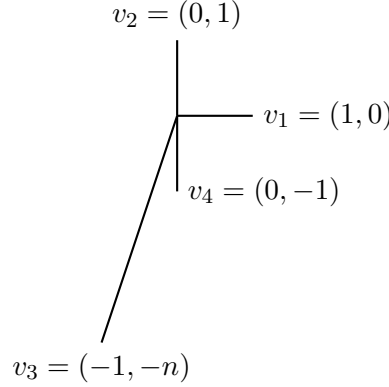
$$v_0 = (0, 1)$$

$$v_1 = (1, 0)$$

$$v_2 = (-2, -3)$$

where again we have only drawn the edges. We leave the rest of this calculation as a nice exercise.


Exercise

Finish Example 3.2.6.

Example 3.2.7. Let's now consider a new space, that will prove very useful to us going forward. Consider 2D fan given by $v_1 = (1,0)$, $v_2 = (0,1)$, $v_3 = (-1,-n)$ and $v_4 = (0,-1)$ with $n > 0$. The drawing of the edges is as follows[6]

Now things are little more subtle when we ask the question "what are the 2D cones in this fan?", as we can*not* just take any cone spanned by two edges. Firstly we note that σ_{24} isn't a cone as it doesn't obey $\sigma_{24} \cap (-\sigma_{24}) = \{(0,0)\}$. Besides that, note that if we take both σ_{13} and σ_{14}, which are both well defined cones, their intersection would be σ_{14}, which is not a face in σ_{13}. Similarly we can't have σ_{13} and σ_{34}. Clearly there are different fans we can construct from these vectors, but here we want to consider the fan which contains cones $\{\sigma_{12}, \sigma_{14}, \sigma_{34}, \sigma_{23}\}$, which we try to depict in the following diagram
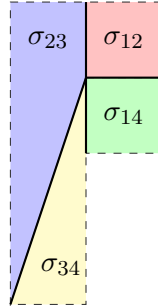
Let's now find the toric variety associated to this fan.

First, from the arguments above, we have that the exceptional set is given by

$$Z(\Sigma) = V(\mathcal{S}_{24}) \cup V(\mathcal{S}_{13}) \cup V(\mathcal{S}_{1234})$$
$$= (z_1, 0, z_3, 0) \cup (0, z_2, 0, z_4) \cup (0,0,0,0)$$
$$= (z_1, 0, z_3, 0) \cup (0, z_2, 0, z_4)$$

where the last line follows from $(0,0,0,0) \in (z_1, 0, z_3, 0) \cup (0, z_2, 0, z_4)$, and the notation on the first line is hopefully clear.

Now we just need to find the group. We have the mapping

$$\phi : (t_1, t_2, t_3, t_4) \mapsto \left(t_1 t_3^{-1}, t_2 t_3^{-n} t_4^{-1}\right),$$

---

[6]For anyone interested, I have used $n = 3$ in drawing this. Of course this is just for illustrative purposes so it doesn't really matter what value we pick.

and so G is given by $t_1 = t_3$ and $t_2 = t_3^n t_4$, or in other words $G \cong (\mathbb{C}^*)^2$, which we can view as the embedding

$$(\mathbb{C}^*)^2 \hookrightarrow (\mathbb{C}^*)^4$$
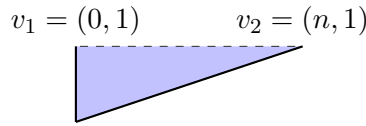$$(t, s) \mapsto (t, t^n s, t, s).$$

So our toric variety is given by

$$X_\Sigma = \frac{\mathbb{C}^4 \setminus \big((z_1, 0, z_3, 0) \cup (0, z_2, 0, z_4)\big)}{(\mathbb{C}^*)^2}.$$

Finally, the dense torus is given exactly by $T = (\mathbb{C}^*)^4 / (\mathbb{C}^*)^2$, where the $(\mathbb{C}^*)^2$ is embedded as written above.

This surface, denoted F_n, is known as the *n-th Hirzebruch surface*.

Example 3.2.8. As a final example consider the fan with edges $(0, 1)$ and $(n, 1)$.

$$v_1 = (0, 1) \qquad v_2 = (n, 1)$$

The exceptional set for this fan is empty, as the only options are $\mathcal{S} \in \{v_1, v_2, \{v_1, v_2\}\}$, all of which correspond to cones in Σ. To be clear, this is *different* to the cases for $\mathbb{CP}^2$ and $\mathbb{WCP}^2_{321}$ which had $Z(\Sigma) = \{(0, 0, 0)\}$. The group action is then found from the map

$$\phi : (t_1, t_2) \mapsto (t_2^n, t_1 t_2)$$

and so G requires $t_2^n = 1$ and $t_1 = t_2^{-1}$. This is just the group $\mathbb{Z}_n$, and so we have

$$X_\Sigma = \frac{\mathbb{C}^2}{\mathbb{Z}_n}.$$

The torus is given by $T = (\mathbb{C}^*)^2 / \mathbb{Z}_n$.

These final two examples are going to prove very useful for us going forward, especially when discussing fibrations and singularity blow ups. For this reason, it is important that these two examples are well understood at this point.

3.2.2 Weightings, Compactness & Singularities

Ok great, hopefully those examples have helped ground the definitions that proceeded them, and we can now go on to discuss how powerful these fan diagrams actually are.

Weightings

The first thing we want to notice is something that was hopefully made suggestively clear from the examples: the group G gives us a quoienting corresponding to scaling(s) of the coordinates directly related to the entries of the vectors. In particular we have

$$[z_1, ..., z_n] \sim \left[\left(\lambda_1^{Q_1^1} ... \lambda_\ell^{Q_\ell^1} \right) z_1, ..., \left(\lambda_1^{Q_1^n} ... \lambda_\beta^{Q_\ell^n} \right) z_n \right], \tag{3.5}$$

where $\lambda_\alpha \in \mathbb{C}^*$ and $\sum_{i=1}^n Q_\alpha^i v_i = 0$ for all $\alpha = 1, ..., \ell$. In particular, for $\mathbb{CP}^2$ we have $\ell = 1$ with $\lambda = t$ and $Q_0 = Q_1 = Q_2 = 1$, i.e. $1 \cdot (0,1) + 1 \cdot (1,0) + 1 \cdot (-1,-1) = 0$. Similarly for $\mathbb{WCP}^2_{321}$ we have $\ell = 1$ and $Q_0 = 3$, $Q_1 = 2$ and $Q_2 = 1$. Then for F_n we have $\ell = 2$ with $\lambda_1 = t$, $\lambda_2 = s$ and $Q_t^1 = 1$, $Q_t^2 = n$, $Q_t^3 = 1$, $Q_t^4 = 0$, $Q_s^1 = 0$, $Q_s^2 = 1$, $Q_s^3 = 0$, and $Q_s^4 = 1$.

It is notationally convenient to display all this information in the form of a *weight system*, which we draw as

$$
\begin{array}{ccc}
z_1 & \cdots & z_n \\
\hline
Q_1^1 & \cdots & Q_1^n \\
& \vdots & \\
Q_\ell^1 & \cdots & Q_\ell^n
\end{array}
$$

So for the $\mathbb{CP}^2$, $\mathbb{WCP}^2_{321}$ and F_n we have

$$
\begin{array}{ccc}
z_0 & z_1 & z_2 \\
\hline
1 & 1 & 1
\end{array}
\qquad
\begin{array}{ccc}
z_0 & z_1 & z_2 \\
\hline
3 & 2 & 1
\end{array}
\quad \text{and} \quad
\begin{array}{cccc}
z_1 & z_2 & z_3 & z_4 \\
\hline
1 & n & 1 & 0 \\
0 & 1 & 0 & 1
\end{array}
$$

We haven't said anything about Example 3.2.8, as this doesn't have a weight system as there is no way to get the v_1 and v_2 to cancel with non-zero Q_i. In a way its weight system vanishes, and so we don't write anything.

Remark 3.2.9. These weight system diagrams are actually incredibly useful as they encode a lot of information. We will add more to them later, but for now notice that from the weight system we not only get the scaling weights, we can use them to reconstruct the generating vectors via $\sum_{i=1}^n Q_a^i v_i = 0$. Further we can immediately read off the dimension of the space – it is simply the number of columns minus the number of rows (excluding the row containing the z_is). This is not hard to see: the number of columns corresponds exactly to the number of coordinates, i.e. the power n factor appearing in the numerator of Equation (3.4), while the number of rows corresponds to how many different scalings we have, which corresponds exactly to the dimension of the group G in the denominator of Equation (3.4). For example, we see straight away that F_n is 2-dimensional from $4 - 2 = 2$. The fact that we can read off the dimension will prove additionally useful later when discussing so-called toric divisors and their linear relations.

Compactness

Next we want to ask the question "is it possible to read off whether the resulting toric variety is compact or not from the fan diagram?" The answer is yes, and is the content of the next proposition.

Proposition 3.2.10. *Let X_Σ be a toric variety associated to a fan Σ. Then X_Σ is compact iff the fan Σ fills $N_\mathbb{R}$.*

The proof of this proposition is easier to see later when considering constructing a fan from a toric variety – Section 3.4 – and so we put off a proof until then. For now we just note that in the examples above, only Example 3.2.8 is non-compact.

Singularities

We now discuss something that is really important for us: the presence of singularities. We note, from the definition Equation (3.4), our toric varieties are orbifolds with potential singularities, depending on what the group G is. We want to see how we can read off whether a toric variety is singular or not, given its associated fan.

The key point is to note the following: consider some fan Σ and form the toric variety X_Σ. Now consider some cone $\sigma \in \Sigma$ and form the toric variety $X_\sigma \subset X_\Sigma$, which we can *define* as the subset obtained by setting $z_\rho = 1$ for all $\rho \in \Sigma(1) \setminus \{\text{edges of } \sigma\}$. We can then patch these X_σ together to give $X_\Sigma = \bigcup_{\sigma \in \Sigma} X_\sigma$. This is just the statement that a fan is given by the union of its cones, and so the associated toric variety is given by the union of the subvarieties.

Now we define $\Sigma_\sigma \subset \Sigma$ to be the fan given by σ and all of its faces (in the sense of a general face, including all edges etc). From the explanation above, we have that $X_{\Sigma_\sigma} = \cup_{\widetilde{\sigma} \in \Sigma_\sigma} X_{\widetilde{\sigma}}$. Putting this together with the fact that there is clearly an injective embedding of $\widetilde{\sigma} \in \Sigma_\sigma$ into σ, simply by definition of Σ_σ, we have that $\cup_{\widetilde{\sigma} \in \Sigma_\sigma} X_{\widetilde{\sigma}} \cong X_\sigma$. We can therefore conclude that $X_{\Sigma_\sigma} \cong X_\sigma$. Finally note that $Z(\Sigma_\sigma) = \emptyset$, simply by the definition of Σ_σ: it contains all the possible cones. We then have the following proposition.

> **Proposition 3.2.11.** *Let Σ be a fan and X_Σ be the associated toric variety. Then X_Σ is smooth (i.e. non-singular) iff every cone $\sigma \in \Sigma$ is generated by vectors which form a $\mathbb{Z}$-basis for $\sigma \cap N$.*

Proof. We show that the basis condition implies smoothness. Consider any top-dimensional cone $\sigma \in \Sigma$, by assumption this is generated by r linearly independent vectors, and so the group G is trivial. This is easiest to see by considering the weight system: there is no way to have these vectors cancel each other and so all $Q_\alpha^i = 0$. Putting this together with $Z(\Sigma_\sigma)$ we have that $X_{\Sigma_\sigma} = \mathbb{C}^r$, and so $X_\sigma \cong \mathbb{C}^r$ which is smooth. Finally putting this together with $X_\Sigma = \cup_{\sigma \in \Sigma} X_\sigma$, we conclude that X_σ is the union of smooth varieties, and so is smooth itself.

The reverse direction, that smoothness implies the basis criteria, is most easily shown using the spectrum approach. As we are not discussing that in these notes, we omit the rest of of the proof.[7] $\square$

For the examples discussed above, we see that:

- $\mathbb{CP}^2$ is smooth, which we know to be true,

- $\mathbb{WCP}^2_{321}$ is singular, which again we know to be true (see the end of the compelx manifold notes),

- F_n is smooth only when $n = 1$, and

- Example 3.2.8 is smooth only when $n = 1$.

So we have a condition for when the toric variety, the obvious question for us to ask is "when does this singularity correspond to an orbifold?" Recalling the definition, this is just the question of "when is G a finite group?".

---

[7]For those unhappy with this, it's not too bad as really we will only be interested in the direction we have shown as our goal is to construct toric varieties from fans.

Proposition 3.2.12. *Let Σ be a fan and X_Σ be the associated toric variety. Then X_Σ is an orbifold iff Σ is simplical.*

Proof. Again we only show the condition $\implies$ orbifold direction. Let $\sigma \in \Sigma$ be a r-dimensional cone, then by definition of a simplical cone it can be generated by r vectors $\{v_1, ..., v_r\}$ which form a basis for the vector space they span. There is therefore only a finite number of ways we can get them to cancel each other, and so G is finite. $\square$

Of course every fan we have considered thus far has been simplical, and so we are dealing with orbifold toric varieties.

3.2.3 Blow Up

We now discuss the important procedure of removing singularities from our toric varieties using the fans. We have just shown that a toric variety is smooth iff all the cones are generated by vectors which form a basis for the intersection of σ with N. So all we have to do to make a singular toric variety smooth is to include more rays such that this condition is met. The formal name for this is *subdividing fans*, which we now explain.

> **Definition.** [Subdividing A Fan] Let Σ be a fan. Then another fan $\widetilde{\Sigma}$ *subdivides* Σ if
>
> (i) $\Sigma(1) \subset \widetilde{\Sigma}(1)$, and
>
> (ii) Each $\widetilde{\sigma} \in \widetilde{\Sigma}$ is contained in some $\sigma \in \Sigma$.

In terms of the toric diagrams, this is very straight forward: we can subdivide a fan Σ by introducing a new ray which "splits" an existing cone into two cones, as the following diagram is meant to indicate.
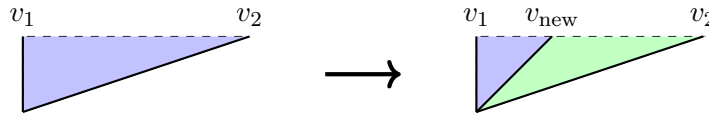
As we explained above, the idea is to take a singular toric variety and subdivide the fan such that we get a smooth result. This procedure is called a *blow up*, the naming of which shall become clear shortly.

The above example is meant to remind us of Example 3.2.8, as we showed that is a singular toric variety. Let's explore the blow up for this fan now.

Example 3.2.13. Recall that we have the fan
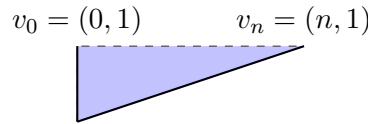
where we have suggestively renamed $v_1 \to v_0$ and $v_2 \to v_n$. We have shown that this is a singular space given by $X_\Sigma = \mathbb{C}^2/\mathbb{Z}_n$. To remove this singularity, we need to introduce

new vectors so that every cone is generated by a basis of $\sigma \cap N$, where N is a 2D lattice. For $v_0 = (0,1)$ the only edge that lies within σ_{0n} that meets this condition is $v_1 = (1,1)$. However we then have the cone σ_{1n} which gives rise to a singularity if $n \neq 2$. However clearly all we have to do is include $v_2 = (2, n)$, and then continue this process until we reach $v_{n-1} = (n-1, 1)$. For $n = 3$ we get the following subdivided fan
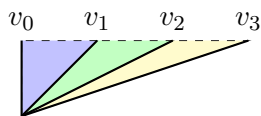
which gives a smooth toric variety with weight system

$$
\begin{array}{cccc}
z_0 & z_1 & z_2 & z_3 \\
1 & -2 & 1 & 0 \\
0 & 1 & -2 & 1
\end{array}
$$

This weight system generalises to the general n case, which follows from

$$v_{n-1} + v_{n+1} = (2, 2n) = 2v_n.$$

For a reason that will become clearer in Section 3.3.5, each of these $(1, -2, 1)$ combinations correspond to a $\mathbb{CP}^1$, so we have a total of $(n-1)$ $\mathbb{CP}^1$s. We can show this in a slightly different way here.

The idea is to consider our space X_Σ as a subvariety in $\mathbb{C}^3$, by considering some well defined polynomial (just as we did when constructing Calabi-Yau manifolds in $\mathbb{CP}^n$ in the complex geometry notes). Here we have

$$[z_0, z_n] \sim [\alpha z_0, \alpha^{-1} z_n] \qquad \text{with} \qquad \alpha^n = 1,$$

so the most general well-defined polynomial is generated by terms

$$z_0^n, \qquad z_n^n \qquad \text{and} \qquad z_0 z_n.$$

Let's label these as $x = z_0^n$, $y = z_n^n$ and $w = z_0 z_n$, and consider them as coordinates for a $\mathbb{C}^3$. The our defining polynomial is

$$P = w^n - xy = 0 \subset \mathbb{C}^3,$$

where the minus sign is included for later convenience. Now it is a fact that a orbifold defined by a zero locus of a polynomial is singular only at points where $P = \partial_i P = 0$, where the derivatives are w.r.t. the coordinates, of course. For our P we see straight away that our orbifold is smooth everywhere apart from at $(x, y, z) = (0, 0, 0)$. This is clear from the fact that we have a $\mathbb{Z}_n$ invariance: the origin is the only point mapped to itself under $\mathbb{Z}_n$.

Ok so we want to remove this singularity, how do we do this? The idea is to slightly deform our defining polynomial such that $(0, 0, 0)$ is no longer singular. Before doing that, we redefine (x, y, w)[8] so that
$$P = w^n - x^2 - y^2,$$

---

[8]I.e. define $\widetilde{x} = \widetilde{x}(x, y, w)$, $\widetilde{y} = \widetilde{y}(x, y, w)$ and $\widetilde{w} = \widetilde{w}(x, y, w)$ to give this form and then relabel $\widetilde{x} \to x$ etc.

for a reason that will become clear in a second. Now, if we deform P by the simple addition of some non-zero constant ϵ,

$$P \to P_\epsilon = w^n - x^2 - y^2 - \epsilon,$$

then we see that $P_\epsilon|_{0,0,0} = -\epsilon \neq 0$, and so we remove the singularity.

What does the new space X_ϵ look like, though? Well it's a fact that the graph of

$$x^2 + y^2 = \epsilon$$

looks like a cylinder with radius $r = \sqrt{\epsilon}$. So our space X_ϵ, which is defined by $x^2 + y^2 = w^n - \epsilon$, consists of a 2D plane with fibres corresponding to cylinders of radius $w^n - \epsilon$.

There are exactly n points on this plane where the cylinders degenerate, in the sense that the radius vanishes. These are just the roots of $w^n = \epsilon$. Let's consider two of these points and consider a path in the (x, y) plane. If we then look at the path in the fibres (at a given "height"), we get a series of circles of continuously varying radius, with the radius vanishing at exactly the two end points of the curve.
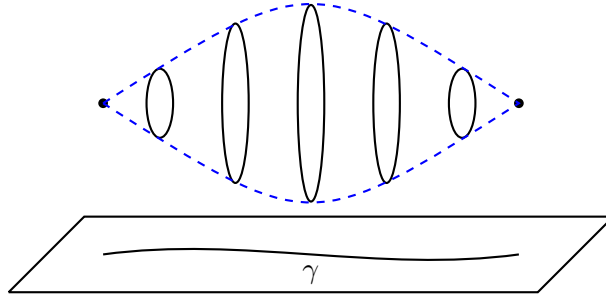
Well this is just a homeomorphic to a 2-sphere (the blue dashed line above), and using the known result $\mathbb{CP}^1 \cong S^2$, we see that we get a total of $(n-1)$ independent $\mathbb{CP}^1$ factors, which is exactly the result we wanted.

We now see more clearly why we call this procedure a blow up: we are taking the n points at which our circle degenerates, and "blowing it up" to finite volume, giving a non-singular space.

Remark 3.2.14. To clarify, it is a *non-trivial* result that this deformation corresponds to the subdivision of the fan explained above. A rough explanation (see Sections 2.6 & 2.6 of [2] for a nice explanation) is as follows: We have a lattice $\Gamma_{3,19}$,[9] and within this lattice we have a 3-plane, Σ. This 3-plane is spanned by the holomorphic $(2,0)$-form, Ω, and the Kähler form ω.[10]

There is a theorem (Theorem 4 in [2]) that says we have a orbifold singularity iff the 3-plane Σ is orthogonal to the so-called *roots* of $\Gamma_{3,19}$. The main "leap-of-faith" we need here is the claims that

(i) The inner product of the root, α, and the Kähler form corresponds to the volume of the $\mathbb{P}^1 \cong S^1$ at our singular points, and

---

[9]For clarity, this lattice turns out to be 22-dimensional and has signature $(3, 19)$, hence the notation.

[10]Note [2] uses J to denote the Kähler form, however I have got used to using J for the complex structure and ω for the Kähler form.

(ii) We have a automorphism group on the 3-plane, so we can freely interchange which directions correspond to Ω and which corresponds to ω in Σ.

So to remove an orbifold singularity, we just need to make sure that none of the roots are orthogonal to Σ, i.e. we need to rotate how Σ sits in $\Gamma_{3,19}$. It follows from our automorphism that we can view this rotation in two ways: the root is non-orthogonal to

(i) $\omega \subset \Sigma$: so we give a finite size to the sphere, thereby removing the singularity. This corresponds to the subdivision of the fan.

(ii) $\Omega \in \Sigma$: Note that the holomorphic volume form is intrinsically tied to the complex structure (which determines what we mean by holomorphic vs antiholomorphic), so this rotation of Ω corresponds to changing the complex structure. Then recalling (see section 4.2.1 of the Complex Manifolds notes) that changing the complex structure corresponds to deforming the defining equation — as if we change J then the normal bundle, N_X, changes, and we think of a non-zero section of N_X as a deformation of P. So this corresponds to our deformation discussion.

The important thing to note here is that this relation between blow up and deformation is *directly* linked to our automorphisms, i.e. that we can link the complex structure to the Kähler form. These automorphisms are specific to $K3$ surfaces, and so, in general, blow ups and deformations are *not* equivalent.

3.3 T-Invariant Subvarieties & Toric Divisors

Ok now that we know how to construct toric varieties from fans, and how to check whether the corresponding toric variety is compact and/or singular, we now want to discuss T-invariant subvarieties and toric divisors. This will lead nicely into the construction of Calabi-Yau hypersurfaces in our toric varieties.

3.3.1 T-Invariant Subvarieties

First let's look at our T-invariant subvarieties. These are particularly easy to describe in terms of our homogeneous coordinate description. Let Σ be a fan and X_Σ the associated toric variety. Then consider some $\sigma \in \Sigma$ which has generating vectors $\{v_1, ..., v_k\}$. We can associate a codimension k subvariety of X_Σ to this cone via

$$Z_\sigma := \{z \in X_\Sigma \,|\, z_1 = ... = z_k = 0\}, \tag{3.6}$$

where we see that it is codimension k from the fact that we have k conditions. Now as T acts on X_Σ by multiplication of non-vanishing complex numbers, this subvariety is clearly T-invariant. Then note that if we have two cones $\sigma, \widetilde{\sigma} \in \Sigma$ where the generating vectors of $\widetilde{\sigma}$ are contained within those for σ (i.e. $\widetilde{\sigma}$ is a face of σ), then the order of inclusion is flipped for the T-invariant subvarieties, i.e. $Z_\sigma \subset Z_{\widetilde{\sigma}}$. The claim is that these are the *only* types of T-invariant subvarieities. Putting this together with the fact that if the cone is *not* in the fan then Z_σ would correspond to an element of the exceptional set $Z(\Sigma)$, and so the subvariety would be empty, we have the following Lemma.

Lemma 3.3.1. *There is a one-to-one correspondance between non-empty T-invariant sub-varieites and cones in fan, given by the ordering reversing mapping $\sigma \mapsto Z_\sigma$.*

It is interesting to note that each Z_σ is in fact a toric variety, and we can construct the lattice and fan from the lattice N and fan Σ for x_Σ: simply take the quotient of N by the sublattice $\sigma \cap N$, and then project every cone in Σ which contains σ as a face onto $\widetilde{N} = N/(\sigma \cap N)$.

Example 3.3.2. As an example, we can construct the T-invariant subvarieties of $\mathbb{CP}^2$ given in Example 3.2.5. We list them below

σ	Z_σ
$\{0\}$	$\mathbb{CP}^2$
$\{(0,1)\}$	$z_0 = 0$
$\{(1,0)\}$	$z_1 = 0$
$\{(-1,-1)\}$	$z_2 = 0$
$\{(1,0),(-1,-1)\}$	$[1:0:0]$
$\{(0,1),(-1,-1)\}$	$[0:1:0]$
$\{(1,0),(0,1)\}$	$[0:0:1]$

which we can see obeys the order reversing inclusion, e.g. $\{(0,1)\} \subset \{(0,1),(-1,-1)\}$ and $[0,1,0] \subset z_0 = 0$.

> **Exercise**
>
> Construct the T-invariant subvarieties for the remaining Examples 3.2.6 to 3.2.8, and check they obey the order reversing condition.

3.3.2 Toric Divisors

The important case of Lemma 3.3.1 for us is that each one-dimensional cone corresponds to a hypersurface in X_Σ. That is,

> we have a one-to-one correspondance between edges and toric divisors.

In what follows we shall denote the toric divisor corresponding to z_i as D_i.

Remark 3.3.3. We call the divisor associated to a blow up, i.e. D_{new}, an *exceptional divisor*. We can think of exceptional divisors as hypersufaces which get "squashed" somehow. We will give a more concrete definition of exceptional divisors after introducing birational equivalence later.

Now recall that Lemma 2.2.3 tells us that to each divisor we can associate some form of line bundle. For our toric divisors, these correspond to the hyperplane line bundles $\mathcal{O}(D_i)$. We can, of course, take a formal sum of our toric divisors to form some new divisor, i.e.

$$D = \sum_{i=1}^{n} a_i D_i.$$

Let's see what happens when we consider the case $a_i = \langle v_i, m \rangle$ for some $m \in M$, with M being the dual lattice to N. Then consider some monomial $z_1^{a_1}...z_n^{a_n}$, which is a section of $\mathcal{O}(\sum_i a_i D_i)$. Our equivalence relation Equation (3.5) then says that (just considering one α value for simplicity) our monomial is equivalent to

$$\left(\lambda^{Q^1} z_1 \right)^{\langle v_1, m \rangle} ... \left(\lambda^{Q^n} z_n \right)^{\langle v_n, m \rangle} = \lambda^{\langle \sum_{i=1}^n Q^i v_i, m \rangle} z_1^{\langle v_1, m \rangle} ... z_n^{\langle v_n, m \rangle}.$$

Then recalling that $\sum_{i=1}^n Q^i v_i = 0$, we see that this monomial is completely invariant under this scaling. This tells us that the monomial is globally well defined on X_Σ, and so corresponds to a *globally* defined memromorphic section, and so it must correspond to a section in a trivial line bundle. That is we *must* have

$$\sum_{i=1}^n \langle v_i, m \rangle D_i \sim 0 \qquad \forall m \in M \tag{3.7}$$

This gives us a set of linear relations between the divisors. It follows from the fact that $\dim M = \dim N$, that we have $\dim N$ such linear relations between our toric divisors, i.e. m has coordinates $m = (m_1, ..., m_{\dim N})$ and we can consider the linear relations given by $m_1 = (1, 0, ..., 0)$, $m_2 = (0, 1, 0, ..., 0)$ etc. This gives us exactly $\dim N$ expressions. So in total we have $|\Sigma(1)| - \dim N$ linearly independent toric divisors. We shall return to this shortly.

3.3.3 Calabi-Yau Condition

We now want to ask the question of "can we read off whether a fan corresponds to a Calabi-Yau space or not?" The answer is yes, let's now see how.

Recall (e.g. from the Complex Manifolds notes) that a Calabi-Yau manifold can be defined as a Kähler manifold that has trivial canonical bundle. The canonical bundle is the line bundle given by the top-dimensional exterior power of the cotangent bundle, i.e. it is the generalisation of the determinant bundle. Next, it follows from the fact that a polynomial of degree k in the coordinates can be identified with the hyperplane bundle $\mathcal{O}(k)$, that we can also identity this with the k-th power of the tangent bundle: i.e. if a polynomial of degree k exists, we can take k derivatives of it. The cotangent bundle is dual to this, and so it follows that the canonical bundle is given by $\mathcal{O}(-\sum_i D_i)$, where we have to set all $a_i = 1$ by antisymmetry.

So the Calabi-Yau condition becomes the requirement that

$$\mathcal{O}\left(-\sum_i D_i \right) \sim \mathcal{O}(0) \qquad \Longleftrightarrow \qquad -\sum_i D_i \sim 0.$$

Putting this together with Equation (3.7), we see that our Calabi-Yau condition can be written by the requirement that there exists some $m \in M_\mathbb{R}$ such that $\langle v_i, m \rangle = -1$[11] for all i. This gives us the following proposition.

---

[11]Of course we can drop this minus sign here, however we include it as it will provide a neat link later when considering polytopes.

Proposition 3.3.4. *Let X_Σ be a toric variety associated to some fan Σ. Then X_Σ is Calabi-Yau iff either of the following, equivalent, conditions apply*

(i) *All the generating vectors end on the same affine hyperplane in $N_\mathbb{R}$; or*

(ii) *The weights Q_a^i obey $\sum_i Q_a^i = 0$ for all a.*

Proof. (i) We can define a hyperplane in $N_\mathbb{R}$ precisely by the condition

$$H_N = \{w_i \in N_\mathbb{R} \,|\, \langle w_i, m \rangle = a\}$$

for some fixed $m \in M_\mathbb{R}$ and $a \in \mathbb{R}$. So the condition $\langle v_i, m \rangle = -1$, where $v_i \in \Sigma(1)$ defines a hyperplane in $N_\mathbb{R}$ that all the generating vectors end on.

(ii) This follows simply from $\sum_i \langle v_i, m \rangle Q_a^1 = 0$ along with $\langle v_i, m \rangle = -1$.

$\square$

Recalling Proposition 3.2.10, we then have the immediate, important, corollary.

Corollary 3.3.5. *A toric Calabi-Yau manifold is non-compact.*

3.3.4 Updating The Weight System

We now return to Remark 3.2.9, which told us that the weight system was useful and said it would be useful when considering toric divisors and their linear relations. Let's now see why that is the case, and also see how we can encode information about a defining polynomial into the weight system.

Recall that we write our weight systems as

$$
\begin{array}{ccc}
z_1 & \cdots & z_n \\
\hline
Q_1^1 & \cdots & Q_1^n \\
& \vdots & \\
Q_\ell^1 & \cdots & Q_\ell^n
\end{array}
$$

Now, to each coordinate z_i we have an associated toric divisor D_i, and so we can think of the columns as representing these toric divisors, i.e. we edit the weight system to look like

$$
\begin{array}{ccc}
z_1 & \cdots & z_n \\
\hline
Q_1^1 & \cdots & Q_1^n \\
& \vdots & \\
Q_\ell^1 & \cdots & Q_\ell^n \\
\hline
\uparrow & & \uparrow \\
D_1 & \cdots & D_n
\end{array}
$$

Now, recall that we showed under Equation (3.7) that the number of linearly independent toric divisors is given by $|\Sigma(1)| - \dim N$. Putting this together with the fact we showed in Remark 3.2.9, that $\dim N =$ (number of columns) $-$ (number of rows), and the fact that $|\Sigma(1)| =$ (number of columns), we immediately conclude that the number of linearly independent toric varieties is given by the number of rows. We can label these independent divisors H_j, and add them to our weight system as

$$
\begin{array}{c|ccc}
 & z_1 & \cdots & z_n \\
\hline
H_1 \to & Q_1^1 & \cdots & Q_1^n \\
 & \vdots & \vdots & \\
H_\ell \to & Q_\ell^1 & \cdots & Q_\ell^n \\
\hline
 & \uparrow & & \uparrow \\
 & D_1 & \cdots & D_n
\end{array}
$$

Indeed we can write the D_is in terms of the H_js using the weights, i.e.

$$
D_i = \sum_{j=1}^{\ell} Q_j^i H_j.
$$

This will become more clear when we look at examples in Section 3.3.6.

Next we note that a hypersurface in our toric variety is given exactly by a divisor, i.e. we can express the defining polynomial, P, as a divisor. Whatever this divisor is, it can be related to our H_js by some given weights p_j. We can add this polynomial to our weight system too as

$$
\begin{array}{c|ccc|c}
 & z_1 & \cdots & z_n & P \\
\hline
H_1 \to & Q_1^1 & \cdots & Q_1^n & p_1 \\
 & \vdots & \vdots & & \vdots \\
H_\ell \to & Q_\ell^1 & \cdots & Q_\ell^n & p_\ell \\
\hline
 & \uparrow & & \uparrow & \\
 & D_1 & \cdots & D_n &
\end{array}
\tag{3.8}
$$

Finally, we recall from the Complex Manifolds notes, that the total Chern class of a hypersurface space, X, in some ambient space, A, is given by

$$
c(X) = \frac{c(A)}{c(P)},
$$

where P is the defining polynomial. Now recall from Section 2.2.2 that we have

$$
c(A) = \prod_{i=1}^{n}(1 + D_i) \qquad \text{and} \qquad c(P) = 1 + \sum_{j=1}^{\ell} p_j H_j,
\tag{3.9}
$$

which we shall verify when considering examples below. In particular, we notice that if want a Calabi-Yau hypersurface we require

$$p_j = \sum_{i=1}^{n} Q_j^i, \tag{3.10}$$

the proof of which is the content of the next exercise.

Exercise

Given Equation (3.9), show that the first Chern class of X is given by

$$c_1(X) = \sum_{j=1}^{\ell} \left(\sum_{i=1}^{n} Q_j^i - p_j \right) H_j,$$

and hence prove the above Calabi-Yau condition.

3.3.5 Intersection Numbers & Fibration Structure

Before going on to study some examples to ground all this information, first we want to discuss intersection numbers between the divisors and how this relates to the fibration structure of our toric variety.

Intersection Numbers

We have shown that each homogeneous coordinate gives rise to a toric divisor, which is a hypersurface in X_Σ. We now want to ask the question "do these hypersurfaces intersect each other, and do they intersect themselves?" The answer to the latter comes from answering the former and then using Equation (3.7).

So how do we know if two toric divisors intersect each other? With some thought, it is clear that this happens only when the corresponding vectors form a cone in the fan. The most intuitive way to see this is probably just the fact that if they generate a cone in Σ, then by Lemma 3.3.1 they form a codimension 2 subvariety. This subvariety is formed exactly as the intersection of the 2 toric divisor hypersurfaces, which follows immediately from Equation (3.6). From here, we use Equation (3.7) to write the self intersection D_i^2 as $D_i \cdot (-a_j)D_j$ for $j \neq i$.

In order to be able to work out the self intersection numbers, we obviously need to know what the $D_i \cdot D_j$ with $j \neq i$ are. In fact the general k-point intersection $D_i \cdot D_j \cdot ... \cdot D_k$ plays an important role for us. Why? Well recall from the Complex Manifolds notes that when we wanted to compute the Euler characteristic we first found the top Chern class and integrated that over the space. Before we always just quoted the result that, for $\mathbb{CP}^n$, $\int D^n = 1$, or if we were considering $\mathbb{CP}^{n_1} \oplus ... \oplus \mathbb{CP}^{n_m}$ that $\int D_1^{n_1}...D_m^{n_m} = 1$. However we already noted at the end of those notes that when you start considering weighted projective spaces you need to be more careful as you get fractional results. So what is going on?

Well $D_i \cdot D_j$ is the intersection of the corresponding hyperplanes. So if $D_i \cdot D_j = 0$ they don't intersect, while if $D_i \cdot D_j = 1$ they intersect exactly once. We then generalise this to the case of n hypersurfaces intersecting. The key point is that if they only intersect once the

resulting intersection space is smooth. So we conclude that $D_i \cdot D_j ... \cdot D_k = 1$ if the generating vectors $\{v_i, v_j, ..., v_k\}$ form a basis of a lattice, as per Proposition 3.2.11. This is precisely why we always took $\int D_i^n = 1$, which we shall clarify shortly.

Now note that we also have that $D_i \cdot D_j ... \cdot D_j = 0$ if the span of the generating vectors don't span a cone in Σ. That is, if $\{v_i, v_j, ..., v_k\}$ don't span a cone in Σ then, by definition of the exceptional set, their common zero locus (which is exactly the intersection of the divisors) is removed from the toric variety X_Σ. So the intersection does not contribute to the integral over X_Σ, or any subspace, i.e. we require $D_i \cdot D_j \cdot ... \cdot D_k = 0$.

We shall talk about our examples shortly to help clarify the details, but first let's discuss fibration structure, so that we can discuss both in the examples.

Fibration Structure

Looking at the toric diagrams, it is tempting to start thinking of the lines as representing surfaces and so the diagram looks like a fibration over some base space. That is the diagram for F_n almost looks like two copies of the diagram of $\mathbb{CP}^1$ at right angles to each other, the only problem being that v_3 doesn't lie in the same plane at v_1. This intuition can actually be made more formal and does indeed lead to a discussion of fibration structures for our toric varieties. The important piece of information we need is that of a morphism between fans.

> **Definition.** [Fan Morphism] Let $\Sigma / \widetilde{\Sigma}$ be a fans in lattices $N_\mathbb{R} / \widetilde{N}_\mathbb{R}$, respectively. Then a *fan morphism* from Σ to $\widetilde{\Sigma}$ is a homomorphism $\psi : N \to \widetilde{N}$, such that for every cone $\sigma \in \Sigma$ the image under $\psi \otimes \mathbb{R}$ is contained in some cone of $\widetilde{\Sigma}$. We say that ψ is *compatible* with Σ and $\widetilde{\Sigma}$.

Our fan morphisms are, as the name states, morphisms between *fans*. Of course what is of more interset to us is how these relate to morphism between the toric *varieties* associated to the fans, i.e. can $\psi : N \to \widetilde{N}$ give/tell us anything about a map $\phi : X_\Sigma \to X_{\widetilde{\Sigma}}$? The answer is "yes", and in order to explore it, we need the notion of a toric morphism, which is defined as we might expect.[12]

> **Definition.** [Toric Morphism] Let $\Sigma / \widetilde{\Sigma}$ be a fans in lattices $N_\mathbb{R} / \widetilde{N}_\mathbb{R}$, respectively, and let $X_\Sigma / X_{\widetilde{\Sigma}}$ be the associated toric varieties. Then a morphism $\phi : X_\Sigma \to X_{\widetilde{\Sigma}}$ is called *toric* if $\phi(T) \subseteq \widetilde{T}$ (i.e. ϕ maps the torus in X_Σ into the torus in $X_{\widetilde{\Sigma}}$), and $\phi|_T$ is a group homomorphism.

We then have the following theorem.

Theorem 3.3.6. *Let $\Sigma / \widetilde{\Sigma}$ be a fans in lattices $N_\mathbb{R} / \widetilde{N}_\mathbb{R}$, respectively.*

(i) *Let $\psi : \Sigma \to \widetilde{\Sigma}$ be compatible with Σ and $\widetilde{\Sigma}$. Then there exists a toric morphism $\phi : X_\Sigma \to X_{\widetilde{\Sigma}}$ such that*

$$\phi|_T = \psi \otimes 1 : N \otimes \mathbb{C}^* \to \widetilde{N} \otimes \mathbb{C}^*.$$

---

[12]We, of course, present this definition in terms of a toric variety coming from a fan, but the key when defining *any* morphism is to ask yourself "what is the defining structure, and how to a preserve it under a map?" This almost always gives you the correct answer. Here the defining structure is the dense torus, and so we just want ϕ to preserve this structure.

(ii) *Conversely, if we have a toric morphism $\phi : X_\Sigma \to X_{\widetilde{\Sigma}}$, then we have a $\psi : N \to \widetilde{N}$ which is compatible with Σ and $\widetilde{\Sigma}$.*

Proof. Omitted. See Theorem 3.3.4 of [3] for more details. $\qquad \square$

We now return to Remark 3.2.3, which said that the toric variety X_Σ actually depends on the lattice N which Σ sits on. The idea is if $\widetilde{N} \subseteq N$ is a sublattice of finite index, since $N_\mathbb{R} = \widetilde{N}_\mathbb{R}$, any fan $\Sigma \subseteq N_\mathbb{R}$ can be viewed as a fan lying on $\widetilde{N}$. The inclusion mapping $\iota : \widetilde{N} \hookrightarrow N$ is compatible with Σ in $\widetilde{N}$ and N, and so gives a toric morphism, the specific manifestation of which is given in the next proposition.

Proposition 3.3.7. *Let $\iota : \widetilde{N} \hookrightarrow N$ be an inclusion mapping of a sublattice of finite index $\widetilde{N} \subseteq N$. Then given a fan $\Sigma \subseteq N_\mathbb{R} = \widetilde{N}_\mathbb{R}$, then we have the toric morphism*

$$\phi : X_{\Sigma,\widetilde{N}} \to X_{\Sigma,N}$$

which presents $X_{\Sigma,N}$ as the quotient $X_{\Sigma,\widetilde{N}}/H$, where $H = N/\widetilde{N}$.

Proof. Omitted. See proposition 1.3.18 and 3.3.7 of [3]. $\qquad \square$

We then have the following useful proposition.

Proposition 3.3.8. *Let $\widetilde{N} \subseteq N$ be a sublattice with $\dim N_\mathbb{R} = n$ and $\dim \widetilde{N}_\mathbb{R} = k$. Then let $\widetilde{\Sigma} \subseteq \widetilde{N}_\mathbb{R}$ be a fan. As $\widetilde{N}_\mathbb{R} \subseteq N_\mathbb{R}$, we can regard $\widetilde{\Sigma}$ a fan in $N_\mathbb{R}$, in which case we shall denote it Σ. Then*

(i) *If $\widetilde{N}$ is spanned by a subbasis of N, then there is an isomorphism*

$$\phi : X_{\Sigma,N} \cong X_{\Sigma,\widetilde{N}} \times T_{N/\widetilde{N}} \cong X_{\Sigma,\widetilde{N}} \times (C^*)^{n-k},$$

where the last line follows simply from the definition of an algebraic torus.

(ii) *Let $N' \subseteq N$ be a sublattice of finite index given by completing a basis for $\widetilde{N}$. Then $X_{\Sigma,N}$ is isomorphic to the quotient of*

$$X_{\Sigma,N'} \cong X_{\Sigma,\widetilde{N}} \times T_{N'/\widetilde{N}} \cong X_{\Sigma,\widetilde{N}} \times (C^*)^{n-k}$$

by the finite abelian group N/N'.

Proof. (i) If $\widetilde{N}$ is spanned by a subbasis of N, then we have $N = \widetilde{N} \times N'$, where N' the $(n - k)$-dimensional sublattice needed to complete the basis for N, i.e. $N' = N/\widetilde{N}$. We then have that $\Sigma = \widetilde{\Sigma} \times \Sigma'$, where $\Sigma' = \{0\} \in N'_\mathbb{R}$ is the trivial fan. Next we note that the toric variety $X_{\Sigma',N'} \cong T_{N'} \cong (\mathbb{C}^*)^{n-k}$. Finally, the result follows from Proposition 3.2.4:

$$X_{\Sigma,N} \cong X_{\widetilde{\Sigma},\widetilde{N}} \times X_{\Sigma',N'} \cong X_{\widetilde{\Sigma},\widetilde{N}} \times T_{N/\widetilde{N}} \cong X_{\widetilde{\Sigma},\widetilde{N}} \times (C^*)^{n-k}.$$

(ii) This follows simply from part (i) and Proposition 3.3.7.

$\qquad \square$

This is hopefully starting to look a bit "fibration-y", i.e. we have that our toric variety is given by the product of two spaces. However this seems like a global property, but fibrations are a local thing. So we need to do a bit more work yet.

The next thing we need to introduce, is the notion of *splitting* a fan. This follows the usual idea from exact sequences.

Definition. [Split Fan] Let N and $\widetilde{N}$ be two lattices and let $\psi : N \to \widetilde{N}$ be a linear, surjective mapping. Then define $N_0 := \ker \psi$ so that we have the short exact sequence of lattices
$$0 \longrightarrow N_0 \longrightarrow N \longrightarrow \widetilde{N} \longrightarrow 0.$$
Now suppose we have cones $\Sigma \subseteq N_{\mathbb{R}}$ and $\widetilde{\Sigma} \subseteq \widetilde{N}_{\mathbb{R}}$ that are compatible with the ψ, and define $\Sigma_0 := \{\sigma \in \Sigma \,|\, \sigma \in (N_0)_{\mathbb{R}}\}$.

Then we say Σ is *split* by $\widetilde{\Sigma}$ and Σ_0 is there exists a subfan $\Sigma' \subseteq \Sigma$ such that

(i) $\psi_{\mathbb{R}} : N_{\mathbb{R}} \to \widetilde{N}_{\mathbb{R}}$ maps every cone $\sigma' \in \Sigma'$ bijectively into a cone $\widetilde{\sigma} \in \widetilde{\Sigma}$ such that $\sigma' \mapsto \widetilde{\sigma}$ defines a bijection $\Sigma' \to \widetilde{\Sigma}$.

(ii) The sum $\sigma' + \sigma_0 \in \Sigma$ for any $\sigma' \in \Sigma'$ and $\sigma_0 \in \Sigma_0$, and conversely that any $\sigma \in \Sigma$ can be written this way.

Split fans are useful to us for the following reason. Our lattice morphism $\psi : N \to \widetilde{N}$ induces a fan morphism
$$\phi : X_{\Sigma,N} \to X_{\Sigma,\widetilde{N}}.$$
Then it follows from Proposition 3.3.8, and the fact that $\widetilde{N} = N/N_0$, that
$$X_{\Sigma_0,N} \cong X_{\Sigma_0,N_0} \times T_{\widetilde{N}}.$$
Next using the fact that $\psi : N_0 \to \widetilde{N}$ is compatible with Σ_0 and the trivial fan in $\widetilde{N}$. This induces the toric morphism
$$\phi|_{X_{\Sigma_0,N}} : X_{\Sigma_0,N} \to T_{\widetilde{N}}$$
We then make the following claim that we actually have $\phi^{-1}(T_{\widetilde{N}}) = X_{\Sigma_0,N}$, and so we conclude
$$\phi^{-1}(T_{\widetilde{N}}) \cong X_{\Sigma_0,N_0} \times T_{\widetilde{N}},$$
which is exactly a fibration structure: the patch in $X_{\Sigma,N}$ over $T_{\widetilde{N}}$ is given by $T_{\widetilde{N}}$ times something, in this case X_{Σ_0,N_0}. We have shown that $X_{\Sigma,N}$ has a fibration structure over the dense $T_{\widetilde{N}} \subseteq X_{\widetilde{\Sigma},\widetilde{N}}$. We want to generalise this to a generic point in $X_{\widetilde{\Sigma},\widetilde{N}}$. We note that the above made no use of the actual splitting, i.e. conditions (i) and (ii) in the definition above. The generalisation of this is given by the the following theorem.

Theorem 3.3.9. *Let Σ be split by $\widetilde{\Sigma}$ and Σ_0, as per the above definition. Then $X_{\Sigma,N}$ is a locally trivial fiber bundle over $X_{\widetilde{\Sigma},\widetilde{N}}$ with fibre X_{Σ_0,N_0}. That is, $X_{\widetilde{\Sigma},\widetilde{N}}$ has an affine open cover $\{U_{\widetilde{\sigma}}\}$ such that*
$$\phi^{-1}(U_{\widetilde{\sigma}}) \cong X_{\Sigma_0,N_0} \times U_{\widetilde{\sigma}}.$$

Proof. The key thing to note is that splitting requires $\phi_{\mathbb{R}} : \Sigma' \to \widetilde{\Sigma}$ to be bijective. This is why we have to consider things locally, i.e. we cannot consider the full $X_{\Sigma,N}$ as two cones in Σ might map into the same cone in $\widetilde{\Sigma}$. This will be clearer when we look at the Hirzebruch surface again.

Pick a $\widetilde{\sigma} \in \widetilde{\Sigma}$ and define

$$\Sigma(\widetilde{\sigma}) := \{\sigma \in \Sigma \,|\, \psi_{\mathbb{R}}(\sigma) \subseteq \widetilde{\sigma}\}.$$

This is clearly a fan morphism, and so we have a toric morphism such that $\phi^{-1}(U_{\widetilde{\sigma}}) = X_{\Sigma(\widetilde{\sigma})}$. To complete the proof, then, we just need to show

$$X_{\Sigma(\widetilde{\sigma}),N} \cong X_{\Sigma_0,N_0} \times U_{\widetilde{\sigma}}.$$

Now $\Sigma(\widetilde{\sigma})$ is split by $\Sigma_0 \cap \Sigma(\widetilde{\sigma})$ and $\Sigma' \cap \Sigma(\widetilde{\sigma})$, which means that any cone in the fan corresponding to $U_{\widetilde{\sigma}}$ must be a cone in $\widetilde{\sigma}$, i.e. we are considering $\widetilde{\Sigma}$ to be given by $\widetilde{\sigma}$ and all its faces. So we have $X_{\widetilde{\Sigma},\widetilde{N}} \cong U_{\widetilde{\sigma}}$.

Now the short exact sequence induces the isomorphism $N_0 \times \widetilde{N} \cong N$. We then have, by the bijectivity condition (i), that there exists a map $\nu_{\mathbb{R}} : \widetilde{\sigma} \mapsto \sigma'$ for all $\sigma' \in \Sigma'$. It follows from the definitions that we actually have $\nu_{\mathbb{R}}(\widetilde{\sigma}) \in \Sigma' \cap \Sigma(\widetilde{\sigma})$.

Putting all this together, we have that the isomorphism $(N_0)_{\mathbb{R}} \times \widetilde{N}_{\mathbb{R}} \cong N_{\mathbb{R}}$ takes the product fan $(\Sigma_0, (N_0)_{\mathbb{R}}) \times (\widetilde{\Sigma}, \widetilde{N}_{\mathbb{R}})$ to the fan $(\Sigma(\widetilde{\sigma}), N_{\mathbb{R}})$. Finally, using Proposition 3.2.4, we obtain

$$X_{\Sigma(\widetilde{\sigma}),N} \cong X_{\Sigma_0,N_0} \times X_{\widetilde{\Sigma},\widetilde{N}} \cong X_{\Sigma_0,N_0} \times U_{\widetilde{\sigma}},$$

which completes the proof. $\qquad\square$

Now, that all seemed rather technical and a complete pain to check for a given toric diagram. However with a bit of thought we see that we can almost instantly "see" the result. Let's elaborate.

In terms of the toric diagrams, we can think of our morphism as distorting the cones, e.g., most importantly for us, projecting them onto some hypersurface. The idea is that if the *edges* all project into edges of some other fan, we know our starting toric variety contains the projected toric variety in some form, as we have a toric morphism from the subfan $\Sigma(1)$ to the contained fan.

The obstruction to it being a *splitting* comes precisely by the questions

1. Are there any cones in the starting fan that don't project into cones in the contained fan?

2. Do any two cones map to the same cone?

The first question poses clear problems for the definition of our fan morphism, while the second causes problems for the bijectivity condition. If the answer to both questions is "no", then we have a splitting. In this case, our original toric variety contains this latter variety as a simple product – it is the base space of the fibration.

However, if the edges project nicely but the cones do *not*, we cannot have a splitting, and so then the projected variety is contained within the starting toric variety in some twisted fashion, in particular we will see that it self intersects! This is an important and neat result, so we summarise it in the following table.

Edges Project Into Edges	Cones Project Into Cones	Contained Manner
✓	✓	Product (embedding)
✓	✗	Twisted (self intersecting inclusion)

Finally, before discussing our examples, we want to quickly introduce some definitions and a proposition.

Definition. [Birationally Equivalent] Let X and Y be varieties (need not be toric), and assume X is irreducible. Then a *rational map* is a morphism from a non-empty open subset $U \subseteq X$ into Y, denoted $f : X \dashrightarrow Y$. A rational map is called *birational* if there exists a inverse of f that is rational, $f^{-1} : Y \dashrightarrow X$. We say that X and Y are *birationally equivalent*. A birational map is essentially a isomorphism between open subsets of X and Y.

Definition. [Exceptional Divisor] Let $f : X \dashrightarrow Y$ be a birational map. Then a divisor D, which corresponds to the codimension-1 subvariety $Z \subset X$ is called *exceptional* if $f(Z)$ has at least codimensional-2 in Y.

Remark 3.3.10. We now understand Remark 3.3.3 better: the blow-*down* corresponds exactly to increasing the codimension.

Definition. [Ruled Surface] A surface, i.e. a 2D variety, X is called a *ruled surface* if it is birationally equivalent to $\mathbb{CP}^1 \times C$, for some curve C.

A ruled surface is actually a very intuitive idea, it is simply the motion of a line — the $\mathbb{CP}^1$ moving along C — and so is a surface that can be "ruled" like the lines on a ruler. Note that the curve C is rather arbitrary, and, importantly for us, it can self intersect.

Proposition 3.3.11. *Let $\pi : X \to C$ be a ruled surface. Then a generic fibre $f \subset X$ does not self intersect, i.e. $f^2 = 0$.*

3.3.6 Examples

Ok that was a lot of potentially very confusing wording, let's discuss our examples to help clarify.

Example 3.3.12. We start with $\mathbb{CP}^2$. This corresponds to the fan with generators $v_0 = (0,1)$, $v_1 = (1,0)$, and $v_2 = (-1,-1)$, with diagram
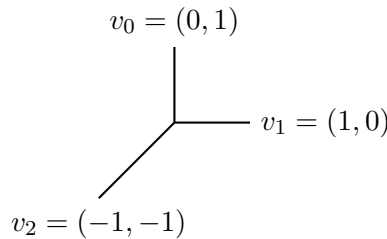
As each pair of vectors spans $\mathbb{Z}^2$, we have

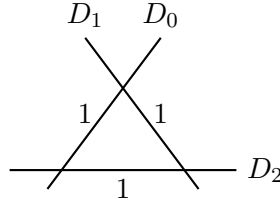$$D_0 \cdot D_1 = D_1 \cdot D_2 = D_0 \cdot D_2 = 1,$$

From Equation (3.7) we then have

$$D_0 - D_2 \sim 0 \qquad \text{and} \qquad D_1 - D_2 \sim 0,$$

where the first equation comes from choosing $m = (0, 1)$ and the second from choosing $(1, 0)$. This gives us

$$D_0^2 = D_0 \cdot D_2 = 1, \qquad D_1^2 = D_1 \cdot D_2 = 1 \qquad \text{and} \qquad D_2^2 = D_2 \cdot D_0 = D_2 \cdot D_1 = 1.$$

We often depict the intersection numbers in a diagram as follows.

where the intersection between divisors is seen by them crossing and the self intersection numbers are indicated by the numbers.

Now, if we consider the two projections corresponding to projecting onto the two entries separately, we see that the edges project onto two copies of the $\mathbb{CP}^1$ diagram. Under both projections σ_{01} projects into a cone of $\mathbb{CP}^1$, i.e. the upper right quadrant projects into a half line on either axis, *however*, under the projection onto the first entry σ_{12} projects onto a full line, which is not contained in a cone of $\mathbb{CP}^1$. Similarly, when we project onto the second entry σ_{02} projects onto a full line. We therefore see that neither of these $\mathbb{CP}^1$ factors are contained in a simple product manner, which is reflected in exactly the fact that all the divisors self intersect.

Note that, as expected, $\mathbb{CP}^2$ is *not* a Calabi-Yau space as the three generating vectors do not end on the same hyperplane. However if we only consider 2 of them, say v_0 and v_1, then we do get a Calabi-Yau space. This is just the statement that if we consider the hypersurface given by $z_2^3 = 0$ then we have a Calabi-Yau space.

The weight system for $\mathbb{CP}^2$ is given by

	z_0	z_1	z_2
$H \rightarrow$	1	1	1
	$\uparrow$	$\uparrow$	$\uparrow$
	D_0	D_1	D_2

from which we can immediately see $D_0 = D_1 = D_2 = H$. These divisors all correspond to the hyperplane line bundle $\mathcal{O}_{\mathbb{CP}^2}(1)$, and we see, from Equation (3.9) that if we want a Calabi-Yau space, we need to consider the divisor $3H$, i.e. a section in $\mathcal{O}_{\mathbb{CP}^2}(3)$, which is exactly the result we got in the Complex Manifolds notes. Equally we have that $\int_{\mathbb{CP}^2} H^2 = 1$, which is also what we used in the previous notes.
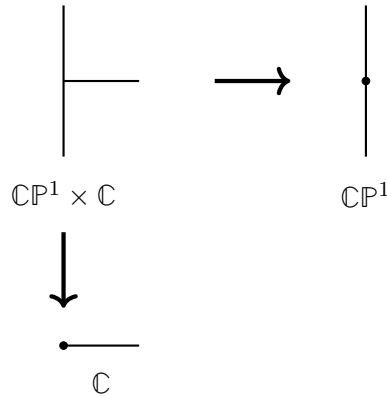
> **Exercise**
>
> Consider the toric diagram for $\mathbb{WCP}^2_{321}$ given in Example 3.2.6, i.e. $v_0 = (0,1)$, $v_1 = (1,0)$ and $v_2 = (-2,-3)$. Write down the weight system for this space, and confirm that a Calabi-Yau space is given by a section in $\mathcal{O}_{\mathbb{WCP}^2_{321}}(6)$, as per the Complex Manifolds notes.

Example 3.3.13. Let's now consider the Hirzebruch surface F_n. This had toric diagram

$$v_2 = (0,1)$$
$$v_1 = (1,0)$$
$$v_4 = (0,-1)$$
$$v_3 = (-1,-n)$$
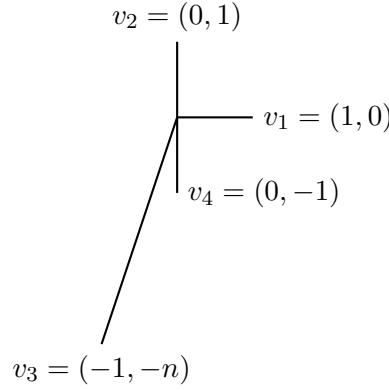
First let's look at the fibration structure. Again all the edges project into edges of $\mathbb{CP}^1$. However now if we project onto the first entry, we see that all cones project onto half lines, and so we do have a $\mathbb{CP}^1$ product factor. In contrast, if we project onto the second factor, the cone σ_{23} projects onto a full line, provided $n \neq 0$, and so our second $\mathbb{CP}^1$ factor is contained in a twisted manner. In the language of split fans, we have that $\Sigma_0 = \{\sigma_2, \sigma_3\} \cong \Sigma_{\mathbb{CP}^1}$ and so our fibres are $\mathbb{CP}^1$, given by (z_2, z_3). In other words, F_n is a non-trivial $\mathbb{CP}^1$-bundle over $\mathbb{CP}^1$. The main contrast to the $\mathbb{CP}^2$ example above is that F_n *is* a $\mathbb{CP}^1$-bundle, it is just non-trivial.

As we have touched upon above, we can think of n as encoding the twisting nature of this $\mathbb{CP}^1$ fibration. We can show this more clearly by showing that *locally* F_n is a trivial $\mathbb{CP}^1$ bundle. We restrict our base space $\mathbb{CP}^1$ (which is the horizontal projection) to the affine open subspace $\mathbb{C} \subset \mathbb{CP}^1$, given by removing the cone generated by (-1).[13] In terms of the toric diagram above, this corresponds to removing the v_3 vector, leaving us with

$$\mathbb{CP}^1 \times \mathbb{C} \qquad\qquad \mathbb{CP}^1$$

$$\mathbb{C}$$

---
[13]Bonus exercise, make sure you follow why this is a $\mathbb{C}$.

where the arrows are meant to indicate the projections, and we have labelled the corresponding toric varieties. This shows us that locally we have a trivial $\mathbb{CP}^1$ fibre over our base space. In other words, we have shown that F_n is a ruled surface, with the curve simply given by $C = \mathbb{C}$.

Ok so now let's look at the intersection numbers. Recalling our fan contains the following cones
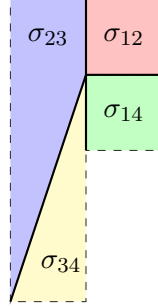
we have smooth cones generated by $\{v_1, v_2\}$ and $\{v_2, v_4\}$, so

$$D_1 \cdot D_2 = D_1 \cdot D_4 = 1,$$

Then from Equation (3.7), with $m = (1,0)$ and $m = (0,1)$ again, we have

$$D_1 - D_3 \sim 0 \qquad \text{and} \qquad D_2 - nD_3 - D_4 \sim 0.$$

It then follows from $D_1 \sim D_3$ and the above intersection relations that
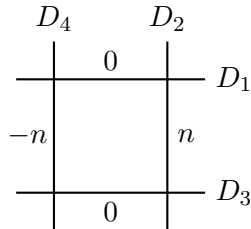
$$D_3 \cdot D_2 = D_3 \cdot D_4 = 1.$$

All other non-self intersections vanish, as $\{v_1, v_3\}$ and $\{v_2, v_4\}$ don't span cones in Σ.

From here we get the self intersection numbers

$$D_1^2 = D_3^2 = 0, \qquad D_2^2 = n \qquad \text{and} \qquad D_4^2 = -n.$$

The important thing to note is the D_1 and D_3 don't self intersect. Note this agrees with Proposition 3.3.11: D_1 and D_3 correspond to setting $z_1 = 0$ and $z_3 = 0$ and so we are in the $[z_2 : z_4]$ line, it then follows from $D_1^2 = D_2^2 = 0$ that our fibres are given by $[z_1 : z_2] \cong \mathbb{CP}^1$. The face that D_2 and D_4 self intersect proportionally to n encodes exactly the non-trivial nature of the $\mathbb{CP}^1$-bundle: the base space is self intersecting. We have the following intersection diagram

The weight system for F_n can be written[14]

---

	z_1	z_2	z_3	z_4
$H_1 \to$	0	1	0	1
$H_2 \to$	1	n	1	0
	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$
	D_1	D_2	D_3	D_4

from which we can clearly see that $D_1 \sim D_3$ and $D_2 \sim nD_3 + D_4$. Note that H_2 is actually the fibre. It is also interesting to note that if $n = 1$ then F_1 corresponds exactly to the blow up of $\mathbb{CP}^2$, and D_4 is the exceptional divisor. This translates nicely in terms of the weight systems: if we set $n = 1$ and then "forget about" $D_4 = H_1$, we are simply left with the weight system for $\mathbb{CP}^2$.

Finally note that $\dim F_n = 2$ and so to find the Euler characteristic we need to know the 2-point intersections H_1^2, H_2^2 and $H_1 \cdot H_2$. We can work these out using our $D_i \cdot D_j$ relations. We have

$$0 = D_1^2 = H_2^2, \qquad -n = D_4^2 = H_1^2 \qquad \text{and} \qquad 1 = D_1 \cdot D_4 = H_1 \cdot H_2.$$

If we then compute the top Chern class, we can find the Euler characteristic using these relations.

Remark 3.3.14. Note in the above we got some negative intersection numbers. This seems intuitively strange — what on Earth does it mean for something to intersect negatively? — however we just take this to be a mathematical extension of our intuition. Really what matters for us is how these intersection numbers relate to computations of things like the Euler characteristic.

Remark 3.3.15. We now see why in Example 3.2.13 the $(1, -2, 1)$ factors correspond to a $\mathbb{CP}^1$: if we just project down along with middle vector (the one with weight -2), the cones project into cones of $\mathbb{CP}^1$, and so we have a $\mathbb{CP}^1$ factor.

Exercise

Compute the intersection numbers[a] of the toric divisors for the smooth fan given in Example 3.2.13, i.e. the fan consisting of vectors $\{(0, 1), (1, 1), ..., (n - 1, 1), (n, 1)\}$. You should get minus the Cartan matrix of $SU(n)$.

[a]For potential future reference, we will also sometimes call intersection numbers "inner forms".

3.4 Constructing Fans From Toric Varieties

We have seen how to construct a toric variety from a fan, we now want to ask the question of when/how can/do we construct a fan from a toric variety? The answer to the when can we question is "when X is so-called *normal*". We will not explain what that means here,[15] but just assume this is the case for us.

---

[15]Because I am not 100% sure yet. Note to self: obviously read up on this.

The key idea to constructing a fan from a toric variety relies in our one-to-one correspondance between T-invariant subvarieties and cones in a fan, Lemma 3.3.1. Consider some toric variety X which contains the torus $T \cong (\mathbb{C}^*)^r$. We can form a lattice from this information as $N = \hom(\mathbb{C}^*, T) \cong \mathbb{Z}^r$. This isomorphism is seen easiest by fixing the identification $T = (\mathbb{C}^*)^r$, then we have the isomorphism $\mathbb{Z}^r \cong \hom(\mathbb{C}^*, T)$ given by

$$(a_1, ..., a_r) \mapsto \big(t \mapsto (t^{a_1}, ..., t^{a_r}) \big). \tag{3.11}$$

We call an element $\psi \in N$ a *one parameter subgroup*. Given a $\psi : \mathbb{C}^* \to T$, we can define a map $f : \mathbb{C}^* \to X$ by $f(t) = \psi(t) \cdot \mathbb{1}_T$, where $\mathbb{1}_T$ is the identity element in T. Clearly the image of f is contained within T. Now let's consider the limit point $\lim_{t \to 0} f(t)$. This may or may not lie in X, but let's suppose it does. Then the *orbit closure*

$$Z_\psi := \overline{T \cdot \lim_{t \to 0} f(t)}$$

is a non-empty, T-invariant subvariety of X. Then by Lemma 3.3.1 there must be some cone in some fan corresponding to this T-invariant subvariety.

How do we extract this cone? We simply consider the set of all one parameter subgroups $\psi \in N$ such that Z_ψ exists, and on this set we define an equivalence relation $\psi \sim \widetilde{\psi}$ if $Z_\psi = Z_{\widetilde{\psi}}$. We then fix an equivalence class and consider all the one parameter subgroups in this equivalence class. If we then take the closure of the convex hull in $N_\mathbb{R} = N \otimes \mathbb{R}$, we get a cone. If we then consider the collection of all cones formed this way, we get a fan, and we have an isomorphism $X_\Sigma \cong X$ between the toric variety constructed from this fan and our original toric variety. It is perhaps not easy to see why this is a cone just from this argument, but it will become clearer when we consider the $\mathbb{CP}^2$ example in a moment. First we want to finally prove the compactness condition, Proposition 3.2.10.

Proof. (Of Proposition 3.2.10.) We shall prove the reverse: that if Σ does not fill $N_\mathbb{R}$ then X_Σ is non-compact. Suppose this is the case, i.e.

$$\bigcup_{\sigma \in \Sigma} \sigma \subsetneq N_\mathbb{R}$$

Then there must exist a one parameter subgroup $\psi \in N$ which satisfies $\lim_{t \to 0} \psi(t) \notin X_\Sigma$, as otherwise there would be a cone in the fan corresponding to it. Putting this together with $\psi : \mathbb{C}^* \to T \subset X$, we must conclude that X_Σ is not compact. $\square$

Example 3.4.1. Once again let's consider out $X = \mathbb{CP}^2$ example. We want to construct the fan from the toric variety. We explained in Example 3.1.1 that the torus in this space is given by

$$T = \{[z_0, z_1, z_2] \in \mathbb{CP}^2 \,|\, z_0, z_1, z_2 \neq 0\}.$$

If we use our scaling to set the first entry to 1 we get

$$T = \{(1, t_1, t_2) \,|\, t_1, t_2 \in \mathbb{C}^*\} \cong (\mathbb{C}^*)^2.$$

The torus action on $\mathbb{CP}^2$ is then given by

$$(t_1, t_2) \cdot [z_0 : z_1 : z_2] = [z_0 : t_1 z_1 : t_2 z_2].$$

From Equation (3.11), our one parameter subgroups are given by

$$\psi_{a,b}(t) = (t^a, t^b)$$

where $(a, b) \in \mathbb{Z}^2$. Considering the action above with these one parameter subgroups, we have seven possible one parameter subgroups who's limit point $t \to 0$ lies in $\mathbb{CP}^2$. They are

a, b	$\lim_{t \to 0} \psi_{a,b}(t)$	$Z_{\psi_{a,b}}$
$a > 0,\ b > 0$	$[1 : 0 : 0]$	$\{[1 : 0 : 0]\}$
$a < 0,\ b > a$	$[0 : 1 : 0]$	$\{[0 : 1 : 0]\}$
$b < 0,\ b < a$	$[0 : 0 : 1]$	$\{[0 : 0 : 1]\}$
$a = b < 0$	$[0 : 1 : 1]$	$\{z_0 = 0\}$
$a > 0,\ b = 0$	$[1 : 0 : 1]$	$\{z_1 = 0\}$
$a = 0,\ b > 0$	$[1 : 1 : 0]$	$\{z_2 = 0\}$
$a = b = 0$	$[1 : 1 : 1]$	$\mathbb{CP}^2$

For clarity, let's explain the a few entries (the rest are then hopefully clear)

- If $a > 0$ and $b > 0$ then $\psi_{a,b}(t) = [1 : t^a : t^b]$ and so in the limit $t \to 0$ the second two entries vanish. The closure of this is just $\{[1 : 0 : 0]\}$.

- If $a < 0$ and $b > a$ then we have $[1 : t^a : t^b] = [t^{-a} : 1 : t^{b-a}]$, and so in the limit $t \to 0$ we get $[0 : 1 : 0]$, which has closure $\{[0 : 1 : 0]\}$.

- If we have $a = b < 0$ then we have $[1 : t^a : t^b] = [t^{-a} : 1 : t^{b-a}] = [t^{-a} : 1 : 1]$, and so $\lim_{t \to 0} \psi_{a,b}(t) = [0 : 1 : 1]$, which has closure $\{z_0 = 0\}$.

If we plot these regions on a graph of (a, b) we get
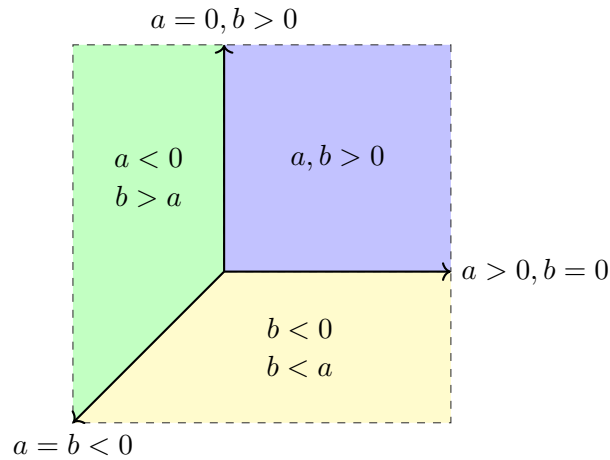
and then the origin corresponds to $a = b = 0$. This is just the toric diagram for $\mathbb{CP}^2$ with the 7 cones we're used to by now. Note, comparing this result to Example 3.3.2, we see that we have the same correspondance between the T-invariant subvarieties and cones in $\Sigma_{\mathbb{CP}^2}$, e.g. $[1 : 0 : 0]$ corresponds to the cone generated by $\{(1, 0), (0, 1)\}$.

3.5 Polytopes

There is a particularly useful way to discuss toric varieties, particularly when discussing mirror symmetry, and it is in terms of *polytopes*.

3.5.1 General Polytopes

First let's forget about everything we've been doing, and in particular forget that we are considering the pair of dual lattices (N, M), and disucss polytopes generally. We will use notation that is suggestive to the cases when we do have lattices, but shall try be clear when we do actually go to these specific cases. The content of this subsection is based off section 2.2 of [3].

> **Definition.** [Polytope] Let $M_\mathbb{R}$ be some real vector space. Consider some set of points $S \subset M_\mathbb{R}$. Then we can define a *polytope* by the convex hull of the set S, i.e.
>
> $$\Delta = \text{Conv}(S) := \left\{ \sum_i \lambda_i m_i \,\Big|\, \sum_i \lambda_i = 1,\ \forall m_i \in S,\ \text{and } \lambda_i \in \mathbb{R}_0^+ \right\} \subseteq M_\mathbb{R}.$$
>
> The dimension of the polytope is equal to the dimensional of the smallest affine subspace in $M_\mathbb{R}$ that contains P.

As it hopefully clear from the definition, polytopes just generalise the idea of polygons in 2D to any dimension, i.e. they are shapes who's boundary is a bunch of points connected with straight lines.

Now, consider the vector space dual to $M_\mathbb{R}$, which we call $N_\mathbb{R}$, and denote their inner product by $\langle \cdot, \cdot \rangle$. We then have the following definition.

> **Definition.** [Polytope Face] Let $(N_\mathbb{R}, M_\mathbb{R})$ be a set of dual vector spaces and let $\Delta \subseteq M_\mathbb{R}$ be a polytope. Then given a non-zero vector $v \in N_\mathbb{R}$ and a $a \in \mathbb{R}$, we can define
>
> $$H_{v,a} := \{ m \in M_\mathbb{R} \,|\, \langle m, v \rangle = a \} \qquad \text{and} \qquad H_{v,a}^+ := \{ m \in M_\mathbb{R} \,|\, \langle m, v \rangle \geq a \}.$$
>
> $H_{v,a}$ is clearly a hypersurface in $M_\mathbb{R}$, and $H_{v,a}^+$ is the upper half plane assocated to this hypersurface. We call a subset $F \subseteq \Delta$ is a *face* of Δ if there exists a $H_{v,b}$ and $H_{v,b}^+$ such that
>
> $$F = H_{v,a} \cap \Delta, \qquad \text{and} \qquad \Delta \subseteq H_{v,a}^+.$$

This definition is hopefully intuitively clear. What we are saying is that we want to consider some hypersurface in $M_\mathbb{R}$ that "touches" Δ, and then the intersection of this hypersurface with Δ is a face of Δ. We call a face of codimension-1 a *facet*, a face of dimension 1 an edge and a face of dimension 0 a vertex. Note we can think of a polytope as the convex hull of its vertices. We give a pictorial example of this for a 2D polytope corresponding to a triangle below.
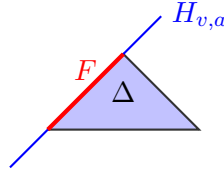
We now note that a polytope Δ is given precisely by the intersection of the a finite number of half planes $H^+_{v_i,a_i}$, i.e.

$$\Delta = \bigcap_{i=1}^{\ell} H^+_{v_i,a_i}$$

is a polytope. It then follows from the definition of H_{v_i,a_i} that the vectors v_i are perpendicular to surfaces H_{v_i,a_i} and point into the intersection, as this is exactly what we need to ensure that $\langle m, v_i \rangle \geq a_i$ for all $m \in \Delta$. We give a pictorial example for a 2D polytope with $\ell = 4$ below.
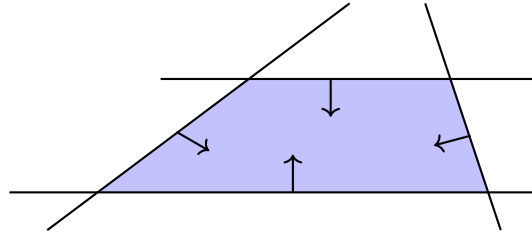
We then have the easily seen results

(i) The full polytope is considered a face of itself;

(ii) Every face is itself a polytope;

(iii) The intersection of two faces is a face.

These results should feel very familiar, indeed they are the same results we have for cones in a fan! Indeed given a polytope, we can define a cone as follows.

> **Definition.** [Polytope Cone] Let $\Delta \subseteq M_{\mathbb{R}}$ be a polytope, and let F denote a generic face. Then we have a cone[16]
> $$\check{\sigma}_F := \bigcup_{r \geq 0} r \cdot (m - f),$$
> for any $m \in \Delta$ and $f \in F$, as above. If we consider all faces F in Δ we get a fan $\check{\Sigma}_\Delta \subseteq M_{\mathbb{R}}$.

The generators of the cone $\check{\sigma}_F$ can be thought of as the linearly independent lines connecting the vertices in $\Delta \setminus F$ to the vertices in F. What will be of most use to us is the cone/fan dual to $\check{\sigma}_F / \check{\Sigma}_\Delta$.

---

[16]We use checked notation as for the work we will do, we like to think of $M_{\mathbb{R}}$ as the dual space, denoting everything with a check.

> **Definition.** [Dual Polytope Cone] Let $\Delta \subseteq M_\mathbb{R}$ be a polytope and let $\check{\Sigma}_\Delta$ be the associated fan. Then, given a $\check{\sigma}_F \in \check{\Sigma}_\Delta$, we define a dual cone $\sigma_F \in N_\mathbb{R}$ via $\langle \sigma_F, \check{\sigma}_F \rangle \geq 0$, i.e.
>
> $$\sigma_F := \{ v \in N_\mathbb{R} \mid \langle f, v \rangle \leq \langle m, v \rangle, \ \forall\, m \in \Delta \text{ and } f \in F \}. \tag{3.12}$$
>
> Again the collection of such cones for all faces gives us a fan $\Sigma_\Delta \subseteq N_\mathbb{R}$.

Now note if the dimension of the fan $\Sigma_\Delta \subseteq N_\mathbb{R}$ is d, then a k-dimensional face $F \subseteq \Delta$ corresponds to a $(d-k)$-dimensional cone in Σ_Δ.

Now comes an interesting, and very useful for us, observation. If $\dim \Delta = \dim M_\mathbb{R}$, then every facet F of Δ has a *unique* supporting hyperplane, which is hopefully clear pictorially — there is no room to "pivot" the $H_{v,a}$ along the facet. We denote the hypersurface and half space by

$$H_F = \{ m \in M_\mathbb{R} \mid \langle m, v_F \rangle = -a_F \} \qquad \text{and} \qquad H_F^+ = \{ m \in M_\mathbb{R} \mid \langle m, v_F \rangle \geq -a_F \}$$

and our polytope is given by

$$\Delta = \bigcap_{F=\text{Facet}} H_F^+ = \{ m \in M_\mathbb{R} \mid \langle m, v_F \rangle \geq -a_F, \ \forall \text{ facets } F \subset \Delta \}, \tag{3.13}$$

where we note that the doublet (v_F, a_F) is unique only up to multiplication by $\lambda \in \mathbb{R}^+$ (which follows from the linearity of $\langle \cdot, \cdot \rangle$). The minus sign appearing in the above expressions is included for later convenience.

3.5.2 Lattice Polytopes & Associated Toric Varieties

Everything in the previous subsection only required $(N_\mathbb{R}, M_\mathbb{R})$ to be a pair of dual *vector spaces*. For clarity, remember that a cone is defined for a vector space, it is just that we have been focusing on the cases when these are actually also given by lattices (N, M). We now do a similar thing for our polytopes, and consider the, creatively named, *lattice polytopes*.

> **Definition.** [Lattice Polytope] Let M be a lattice and let $M_\mathbb{R} := M \otimes \mathbb{R}$. Consider some polytope $\Delta \subseteq M_\mathbb{R}$. Then we call Δ a *lattice polytope* if the vertices of Δ are lattice points in M. We call the ordered set $\Delta \cap M = \{ m_0, ..., m_k \}$ the *characters* of M.

It is important to note that even if $M_\mathbb{R}$ is related to some lattice M, a general polytope $\Delta \subseteq M_\mathbb{R}$ is *not* a lattice polytope. This is completely analogous to the case when we have a cone who's generating vectors are not lattice points. From now on, however, unless otherwise specified, we will always be dealing with lattice polytopes and so simply refer to them as polytopes.

The first thing we notice about such polytopes is that we can always make them top-dimensional (simply by reducing the dimension of M to match), and so Equation (3.13) always applies. If we put this together with the fact that a facet corresponds to a 1-dimesional cone in Σ_Δ, we can alter Equation (3.13) to

$$\Delta = \{m \in M_{\mathbb{R}} \,|\, \langle m, v_F \rangle \geq -a_F, \, \forall v_F \in \Sigma_\Delta(1)\}. \tag{3.14}$$

The fact that we can produce a fan $\Sigma_\Delta \subseteq N_{\mathbb{R}}$ from a polytope $\Delta \subseteq M_{\mathbb{R}}$, means that we can also produce a toric variety from Δ, simply by X_{Σ_Δ} following the material already discussed. There is another way we can view this construction of a toric variety from a polytope, which we now outline.

Consider some toric variety defined by some lattice $N \cong \mathbb{Z}^r$, and interpret M as the dual lattice, i.e. $M = \hom(N, \mathbb{Z})$. Similarly to the previous section, where we showed that we can view N as $\hom(C^*, T)$, we can interpret M as $\hom(T, \mathbb{C}^*)$. Explicitly we have

$$m_i : (t_0, ..., t_r) \mapsto \prod_{j=0}^{r} t_j^{m_{ij}},$$

where m_{ij} are the coordinates of $m_i \in M$, i.e. $m_i = (m_{i0}, ..., m_{ir})$.

In this way, we can interpret the characters of M as nowhere vanishing, holomorphic functions on T. If we have characters $\{m_0, ..., m_k\}$, we can then construct the map

$$f : T \to \mathbb{CP}^k$$
$$t \mapsto f(t) := \big(m_0(t), ..., m_k(t)\big).$$

It follows immediately from $m_i(t) \neq 0$ that such a map is an inclusion, and then from the fact that each m_i is distinct, that we have an embedding.

We note that this embedding is actually the embedding of a k-torus, where the action of T on $\mathbb{CP}^k$ is given by coordinatewise multiplication by $f(t) = (m_0(t), ..., m_k(t))$, i.e. it makes $C\mathbb{P}^k$ a toric variety. We then define $\mathbb{CP}_\Delta \subset \mathbb{CP}^k$ to be the closure of $f(t)$, and note that, by definition, our torus action is closed within $\mathbb{CP}_\Delta$, and so $\mathbb{CP}_\Delta$ is itself a toric variety.

Remark 3.5.1. Note that the construction of $\mathbb{CP}_\Delta$ does _not_ depend on how we defined our characters, i.e. it does not depend on the ordering of $\Delta \cap M$.

Remark 3.5.2. The toric variety constructed in this way is _normal_. A non-normal toric variety can be constructed by considering a subset $\{\widetilde{m}_0, ..., \widetilde{m}_\ell\} \subsetneq \Delta \cap M$ who's convex hull is still the full Δ. Unless otherwise stated, we will only consider normal varities, this remark is just included for completeness.

Proposition 3.5.3. _There exists an isomorphism_ $X_{\Sigma_\Delta} \cong \mathbb{CP}_\Delta$, _given by_

$$(z_1, ..., z_n) \mapsto \left[\prod_{i=1}^{n} z_i^{\langle m_0, v_i \rangle} : ... : \prod_{i=1}^{n} z_i^{\langle m_k, v_i \rangle} \right]$$

where $(z_1, ..., z_n)$ _are the homogeneous coordinates of_ X_{Σ_Δ} _and_ $\{v_1, ..., v_n\} = \Sigma_\Delta(1)$, _the edges of_ Σ_Δ.

Example 3.5.4. The simplest example we can consider is the polytope given by the 2 vertices $\{(1), (-1)\} \subset \mathbb{R}$. The polytope is given by the convex hull of these points, which is simply the line depicted below.
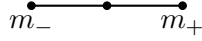
The characters are simply $\{(-1), (0), (1)\} \subset \mathbb{Z}$. This has two proper faces, simply the vertices $m_\pm$, and it is hopefully clear the corresponding fan in N is given by generators $v_- = (1)$ and $v_+ = (-1)$, where the notation means that v_- is the generator corresponding to the face m_-. This gives the fan corresponding to $\mathbb{CP}^1$, and so we see that $X_{\Sigma_\Delta} = \mathbb{CP}^1$. Explicitly, denoting $m_0 = (-1)$, $m_1 = (0)$, $m_2 = +1$, $v_1 = (-1)$, $v_2 = (0)$ and $v_3 = (+1)$, we have

$$(z_1, z_2) \mapsto \left[z_1 z_3^{-1} : 1 : z_1^{-1} z_3 \right]$$

which is an embedding of $\mathbb{CP}^1$ into $\mathbb{CP}^2$.

Example 3.5.5. As a slightly more involved example, let's consider the case of polytope given by vertices $\{(0,0), (1,0), (0,1)\} \subset \mathbb{R}^2$. The full polytope is depicted in the following diagram.
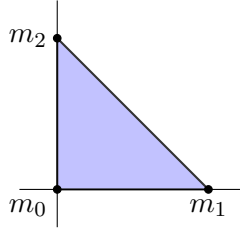
To construct the fan Σ_Δ we construct our σ_F. We have 3 1D faces in Δ just given by the black lines drawn above, i.e. $F_{01} = \{(x,0) \in \mathbb{R}^2 \,|\, 0 \leq 0 \leq 1\}$ and similarly we define F_{02} and F_{12}. It is hopefully clear that the cones associated to these faces are generated by $v_{01} = (0,1)$, $v_{02} = (1,0)$ and $v_{12} = (-1,-1)$, which we now simply label as $v_1 = (1,0)$, $v_2 = (0,1)$ and $v_3 = (-1,-1)$, which we plot below along with the polytope.
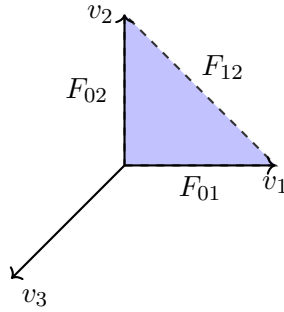
which we recognise as the fan for $\mathbb{CP}^2$. We also note that the rays are normal to the faces they come from, e.g. $v_1 = v_{02}$ is normal to the face F_{02}. This is, of course, exactly the arrows we had pointing inwards from our $H_{v,a}$ hypersurfaces defining Δ. For this reason, we often call Σ_Δ the *normal fan* of Δ.

We can check we have $\mathbb{CP}^2$ using our map:

$$(z_1, z_2, z_3) \mapsto [1 : z_1 z_3^{-1} : z_2 z_3^{-1}] = [z_3 : z_1 : z_2],$$

where the equality follows from scaling by z_3.

> **Exercise**
>
> Considering the polytope with vertices $\{(0,0),(2,0),(0,2)\}$, show that the resulting toric variety is given by $\mathbb{CP}^2$ embedded into $\mathbb{CP}^5$ as
>
> $$(z_1, z_2, z_3) \mapsto [x_1^2 : x_1 x_2 : x_2^2 : x_1 x_3 : x_2 x_3 : x_3^2]. \qquad (3.15)$$
>
> This is known as the *Veronese embedding* of $\mathbb{CP}^2$.
> *Hint: If stuck, see Example 7.9.4 of [4], along with our example above.*

This exercise highlights an important point: if we keep the *shape* of the polytope the same, i.e. we just scale all vertices by the same positive number, then the normal fan, and so resulting toric variety, is unchanged. What changes is how the space manifests itself. In particular notice that every term on the right-hand side of Equation (3.15) is of homogeneous weight 2 in the $\mathbb{CP}^2$ coordinates. We shall expand on this in the next subsection, in particular see Example 3.5.6.

3.5.3 Calabi Yau Hypersurfaces

The above construction is very useful, however as the examples have demonstrated, we get the full ambient toric variety, but we are often interested in hypersurfaces in these spaces in order to construct Calabi Yau spaces. The obvious question to ask is "can we tweak this construction so that we get the hypersurfaces? The answer is yes, and we now flush out the details. This subsection is based off Appendix B of [5] and the interested reader is directed there for more info.

As we have seen, our hypersurfaces are encoded in two pieces of information: the ambient toric variety and the defining polynomial, which, as we explained in the complex manifolds notes, is actually a section in a line bundle $\mathcal{L}$. We can express our polynomial/line bundle in terms of the divisor

$$D = c_1(\mathcal{L}) = \sum_i a_i D_i, \qquad (3.16)$$

where D_i are the toric divisors, which carry weights a_i. So if we want to be able to produce our hypersurface from a polytope Δ, we need to be able to produce both the ambient variety and the divisor D.

Well, note that if we are given a divisor Equation (3.16) and the generating rays of our variety, $\Sigma(1)$, we can produce a polytope $\Delta_P \subset M_{\mathbb{R}}$ via

$$\Delta_P = \{m \in M_{\mathbb{R}} \,|\, \langle m, v_i \rangle \geq -a_i, \ \forall v_i \in \Sigma(1)\}.$$

We call a polytope produced by a polynomial a *Newton polytope*.

So the idea is simply to run this backwards, which we summarise in the following nice box.

> Given a polytope Equation (3.14) (with $a_i \in \mathbb{N}$), we produce the ambient toric variety by constructing the fan Σ_{Δ}, and then we define a hypersurface in this variety by thinking of Δ as the Newton polytope of a line bundle Equation (3.16). The homogeneous degree of the hypersurface is given by $\sum_i a_i$.

An important thing to notice is that if we consider the characters of Δ, we can produce a basis for the group of holomorphic sections in $\mathcal{L}$ by

$$p(m) = \prod_i z_i^{\langle m, v_i \rangle + a_i}, \tag{3.17}$$

which follows from the fact that the numerator is always a positive integer, as $\langle m, v_i \rangle \in \mathbb{N}^{> -a_i}$.

Example 3.5.6. Let's try construct the hypersurface defined by a polynomial of degree k in $\mathbb{CP}^2$. We have already seen that we get the ambient toric variety $\mathbb{CP}^2$ if we consider the polytope with vertices $\{(\lambda, 0), (0, \lambda), (0, 0)\} \subset \mathbb{R}^2$ for some positive λ.

Labelling the generating vectors of $\mathbb{CP}^2$ as before, i.e. $v_1 = (1, 0)$, $v_2 = (0, 1)$ and $v_3 = (-1, -1)$, our polytope is then given by $a_1 = a_2 = 0$ and $a_3 = \ell$. This follows from

$$0 \leq \langle m, (1, 0) \rangle \leq \ell, \qquad 0 \leq \langle m, (0, 1) \rangle \leq \ell \qquad \text{and} \qquad -\ell \leq \langle m, (-1, -1) \rangle \leq 0,$$

for all $m \in \Delta$. Putting this together with the fact that we want $a_1 + a_2 + a_3 = k$, the degree of our polynomial, we see we need $\ell = k$.

This is exactly the situation we had in the exercise at the end of the last subsection (which has $k = 2$), however now we also have Equation (3.17), which allows us to read off basis for holomorphic sections in our line bundle $\mathcal{L} = \mathcal{O}_{\mathbb{CP}^2}(k)$ easily. Explicitly we have

$$p((r, s)) = z_1^{\langle (r,s), (1,0) \rangle} z_2^{\langle (r,s), (0,1) \rangle} z_3^{\langle (r,s), (-1,-1) \rangle + k} = z_1^r z_2^s z_3^{k-r-s},$$

which is always holomorphic as a lattice point in Δ requires $r + s \leq k$. In particular the vertices correspond to

$$p((k, 0)) = z_1^k, \qquad p((0, k)) = z_2^k \qquad \text{and} \qquad p((0, 0)) = z_3^k.$$

Remark 3.5.7. At first it might feel a bit odd that we are setting $a_1 = a_2 = 0$ but saying that we can produce a polynomial of non-vanishing degree in z_1 and z_2, which correspond to D_1 and D_2. However we have to note that although $\mathbb{CP}^2$ does have 3 toric divisors, there are $\dim \mathbb{CP}^2 = 2$ linear relations between these toric divisors, and so we actually only have 1 linearly independent divisor. This linear relation is given exactly by the weights, so here we simply have $D_1 \sim D_2 \sim D_3$, and so the polynomial $z_1^r z_2^s z_3^{k-r-s}$ has divisor

$$rD_1 + sD_2 + (k - r - s)D_3 \sim (r + s + k - r - s)D_3 = kD_3,$$

which is exactly what we had.

Ok great, so we know how to use polytopes to produce hypersurfaces in toric varieties. Of course the case of most interest to us is a polynomial of degree $(n + 1)$ in $\mathbb{CP}^n$. With the above remark in mind, we see that the line bundle for such a polynomial is given by

$$D = \sum_{v_i \in \Sigma(1)} D_i,$$

i.e. we take each toric divisor with weight 1.[17] This means we have set $a_i = 1$ for all i, and so our polytope is

---

[17]Note this will also be true for weighted projective spaces, because now the linear relations are $D_1 = 3D_3$ and $D_2 = 2D_3$ for $\mathbb{CP}^3_{321}$, for example, so the sum is $D = D_1 + D_2 + D_3 = 6D_3$, which is our Calabi-Yau condition.

$$\Delta = \{m \in M_{\mathbb{R}} \,|\, \langle m, v_i \rangle \geq -1 \,\forall v_i \in \Sigma(1)\}. \tag{3.18}$$

As we have tried to be clear to emphasise, in general the vertices of Δ are *not* lattice points in M, i.e. Δ is not a lattice polytope. As the examples above hopefully made clear, if $\Delta \subseteq M_{\mathbb{R}}$ is a lattice polytope, then the vertices of Σ_{Δ} are lattice points in N. We can use these lattice points to define a polytope, $\Delta^{\circ} \subseteq N_{\mathbb{R}}$, simply as their convex hull. We call Δ° the *polar dual* of Δ. The fan over Δ° is, of course, equal to the fan Σ_{Δ}. If we further impose Equation (3.18), we call the pair *reflexive*. It is a necessary condition for reflexivity that the origin is a unique interior point of the polytope.

Reflexive polytope will be of most interest to us, as these allow us to construct Calabi-Yau hypersurfaces from toric varieties that lie on lattice points, which is exactly what we were doing earlier. We get a *specific* Calabi-Yau hypersurface by specifying the defining polynomial, which here corresponds to using Equation (3.17) with $a_i = 1$, and then specifying some complex coefficients α_m to give us the terms we want, then considering the zero locus. That is

$$0 = \sum_{m \in \Delta} \alpha_m p(m) = \sum_{m \in \Delta} \prod_{v_i \in \Sigma(1)} \alpha_m z_i^{\langle m, v_i \rangle + 1}$$

is our defining polynomial.

4 | Elliptic Fibrations

We now want to discuss a particular type of toric varieties, which demonstrate just how powerful and simple this construction is: elliptic fibrations. From a string theory point of view, elliptic fibrations are vital in the "pomotion" of M-theory to F-theory.[1]

4.1 Elliptic K3 Surfaces

The first thing we need to understand is what exactly an elliptic curve is. We work off the following rough definition.

> **Definition.** [Elliptic Curve] An *elliptic curve* is a smooth, sub variety defined by the solutions to
> $$y^2 = x^3 + ax + b, \tag{4.1}$$
> where a, b are complex numbers. The smoothness condition can be expressed by the non-vanishing of the discriminant
> $$\Delta = -(4a^3 + 27b^2).$$

As the title of this chapter suggests, we somehow want to fibre elliptic curves in our toric varieties. The obvious question is "how do we do this?" Well, looking at Equation (4.1) and trying to think of it as the hypersurface in some complex projective space, we see we get something projectively well defined if

$$y \sim \lambda^3 y, \qquad x \sim \lambda^2 x, \qquad a \sim \lambda^4 a \qquad \text{and} \qquad b \sim \lambda^6 b.$$

With a bit more thought, we see that this can be written as a degree 6 polynomial in $\mathbb{WCP}^2_{321}$, i.e. if we use coordinates $[y : x : w] \sim [\lambda^3 y, \lambda^2 x, \lambda w]$, we have an elliptic curve via

$$y^2 = x^3 + fw^4 x + gw^6, \tag{4.2}$$

where f, g are smooth functions. We can, in fact, obtain this equation from the general weight 6 polynomial in $\mathbb{WCP}^2_{321}$. That is, we can start from

$$0 = \alpha y^2 + \beta x^3 + \gamma w^6 + \delta w^2 x^2 + \tau w^4 x + \rho w^3 y + \sigma xyw,$$

and then use standard change of variables techniques (i.e. $y \to \widetilde{y}(x, y, w)$ etc) to eliminate terms to leave us with a $\widetilde{y}^2 = F(\widetilde{x}, \widetilde{w})$ expression. In particular, we can shift y so that the y^2

---

[1]Although I know *very* little about this topic, so shall make no further comments on this in these notes.

term cancels the linear terms in y, and then further shift x so that the x^3 term cancels the w^2x^2 term. This will then give us something of the form Equation (4.2), where f, g depend on $(\alpha, \beta, ...)$.

We now recall that the Calabi-Yau condition for a hypersurface in a weighted projective space is given by $d = \sum w_i$, where d is the degree of the polynomial and w_i are the scaling weights. So for $\mathbb{WCP}^2_{321}$ we need $d = 3 + 2 + 1 = 6$, which is exactly what we have! So we have just constructed a K3-surface[2] as an elliptic curve in $\mathbb{WCP}^2_{321}$. This is known as an *elliptic K3 surface*. We will use these a lot, and so write this again in a nice box.

> The hypersurface given by the zero locus of
>
> $$P = -y^2 + x^3 + fw^4x + gw^6$$
>
> in $\mathbb{WCP}^2_{321}$ is an elliptic curve.

4.2 Fibrations

So far we haven't really made use of our toric variety techniques, i.e. we can obtain an elliptic curve just using the material from the complex manifolds notes. We now come to the second part of the title of this chapter: the fibrations! It is at this point that the toric geometry techniques really take the reigns and make otherwise highly non-trivial constructions much easier to deal with. For example, let's imagine we are asked to construct a Calabi-Yau 3-fold given by an elliptic curve fibred over $\mathbb{CP}^1$. That is, for every point $[z_0 : z_1] \in \mathbb{CP}^1$ we want our defining polynomial to look like Equation (4.1). This is an example of what we call an *elliptic K3 surface*, as it is a K3 surface (that is a 2D Calabi-Yau) containing an elliptic curve.

The obvious thing to try is using our $\mathbb{WCP}^2_{321}$ construction of elliptic curves and somehow add the the $\mathbb{CP}^1$ coordinates in. We could try to consider a simple product space and a defining equation of the form

$$h(z_0, z_1)\big(-y^2 + x^3 + fw^4x^2 + gw^6\big) = 0,$$

where $h(z_0, z_1)$ is a degree 2 polynomial in the $\mathbb{CP}^1$ coordinates. The problem with this is that our elliptic fibration wants the $-y^2 + x^3$ term to have unit coeffient, so here we would have to divide through by $h(z_0, z_1)$, and this might lead to a very bad singularity (as z_0 or z_1 may be zero somewhere)

With some thought, we can see the better option is to consider f and g to be functions of $[z_0 : z_1]$:

$$P = -y^2 + x^3 + f(z_0, z_1)w^4x + g(z_0, z_1)w^6$$

In this case, our singularity problem is reduced again to the requirement that the discriminant $\Delta = -(4f^3 + 27g^2)$ be non-zero. However we now have a new problem of the fact that our defining equation must be projectively well defined and so now the $\mathbb{WCP}^2_{321}$ coordinates must have non-trivial weight under the $[z_0 : z_1]$ scaling. In other words, f and g will now transform non-trivially under the $\mathbb{CP}^1$ scaling, so at least the y^2 and x^3 terms must also transform so

---

[2]For clarity, recall that a K3 surface is a 2D Calabi-Yau space.

that we can factor out the scaling, i.e. $P \to \lambda^m P$, for some $m \in \mathbb{N}$. We then further need to require that our Calabi-Yau condition is met, and so we start getting into potentially very confusing subtleties.

The point we now want to make is that the toric geometry construction makes this computation a lot easier, and essentially a counting exercise. We will start with the elliptic K3 over $\mathbb{CP}^1$ as an introductory example to explain the idea in detail, and then quickly show how we can generalise this to more complicated ideas.

4.2.1 Over $\mathbb{CP}^1$: An Elliptic K3 Surface

We start by recalling the fans of $\mathbb{WCP}^2_{321}$ and $\mathbb{CP}^1$:
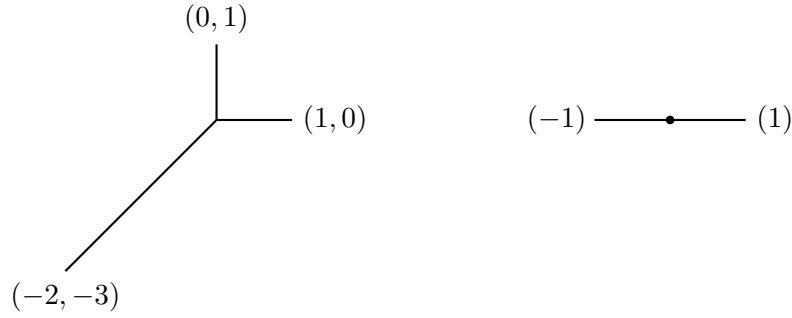
Next we recall that in order to get a fibration, we need to have a fan morphism. Then in order to make the $\mathbb{CP}^1$ our fibre we needed all cones to project into the $\mathbb{CP}^1$ cones. In order to do this, it is hopefully clear that we need a 3D lattice. We take our $\mathbb{CP}^1$ fan morphism to corresponding to projecting onto the third entry.

Now, for $\mathbb{WCP}^2_{321}$ to be a fibre, it follows from Theorem 3.3.9 that the third entries of (v_x, v_y, v_w) *must* be 0, i.e. we need these vectors to project onto the trivial fan of $\mathbb{CP}^1$, which is 0. This is what ensures that every *cone* projects into an cone of $\mathbb{CP}^1$.

Great, all that is left to do is to consider what allowed values for the first two entries of the vectors (v_0, v_1) are. We have the rule that under the projection onto the first two coordinates, every *edge* must project into an edge. It follows simply, then, that the allowed entries are scalar multiples of $(0, 0), (1, 0), (0, 1)$ $(-2, -3)$.

So in total we have that every toric fibration of $\mathbb{WCP}^2_{321}$ over $\mathbb{CP}^1$ is given by vectors

$$(1,0,0),(0,1,0),(-2,-3,0) \begin{cases} (0,0,1),(0,0,-1) \\ (\alpha,0,1),(0,0,-1) \\ (0,\alpha,1),(0,0,-1) \\ (-2\alpha,-3\alpha,1),(0,0,-1) \\ (0,0,1),(\alpha,0,-1) \\ \vdots \\ (\alpha,0,1),(\beta,0,-1) \\ (0,\alpha,1),(\beta,0,-1) \\ \vdots \end{cases} \tag{4.3}$$

where $\alpha, \beta \in \mathbb{N}$, and the ... correspond to all other permutations (which are hopefully easy to see).

Ok great, we have constructed all of the fibrations of $\mathbb{WCP}^2_{321}$ over $\mathbb{P}^1$, we now want to take a hypersurface in this space that gives us an elliptic curve fibration over $\mathbb{CP}^1$. As before we want this to come from the defining equation

$$P = -y^2 + x^3 + f(z_0, z_1)w^4 x + g(z_0, z_1)w^6. \tag{4.4}$$

The questions we want to address is the weightings of each coordinate, and the Calabi-Yau condition. Both of these things are contained in our weight systems, Equation (3.8). So essentially what we need find are the Q^i_j and p_j entries of

y	x	w	z_0	z_1	P
Q^y_1	Q^x_1	Q^w_1	$Q^{z_0}_1$	$Q^{z_1}_1$	p_1
Q^y_2	Q^x_2	Q^w_2	$Q^{z_0}_2$	$Q^{z_1}_2$	p_2

Our two scalings are the $\mathbb{WCP}^3_{321}$ and $\mathbb{CP}^1$ scalings. We shall take the top line, i.e. Q_1, to be the former. This line is straight forward: we don't need z_0 or z_1 to be charged under this scaling and so we simply get a Calabi-Yau polynomial in $\mathbb{WCP}^2_{321}$:

y	x	w	z_0	z_1	P
3	2	1	0	0	6
Q^y_2	Q^x_2	Q^w_2	$Q^{z_0}_2$	$Q^{z_1}_2$	p_2

So we just need to find the second line. In order to obtain a Calabi-Yau from the $\mathbb{CP}^1$ perspective, it follows that $Q^{z_0}_2 = Q^{z_1}_2 = 1$. Then we note that our elliptic fibration equation, Equation (4.4), doesn't require w to be charged here, as there is no term with just w in it. So we can set $Q^w_2 = 0$.

y	x	w	z_0	z_1	P
3	2	1	0	0	6
Q^y_2	Q^x_2	0	1	1	p_2

If we want a Calabi-Yau, we require $p_2 = \sum_j Q^j_2$, as per Equation (3.10), and so we have

$$p_2 = Q^y_2 + Q^x_2 + 2.$$

We then simply use Equation (4.4): it follows from the y^2 and x^3 term that $2Q^y_2 = 3Q^x_2 = p_2$, which we can easily solve to obtain $Q^y_2 = 6$ and $Q^x_2 = 4$. So in total we have

y	x	w	z_0	z_1	P
3	2	1	0	0	6
6	4	0	1	1	12

We then finally apply this same calculation backwards to find the order for f and g, i.e. what powers do $[z_0 : z_1]$ appear as. The Q_1 weighting won't tell us anything as z_0 and z_1 are not charged under this. However, the Q_2 weighting gives us

$$12 = 1 \cdot O_f + 4Q^w_2 + Q^x_2, \qquad \text{and} \qquad 12 = 1 \cdot O_g,$$

which if you plug in the values above give us

$$O_f = 8 \qquad \text{and} \qquad O_g = 12.$$

> **Exercise**
>
> Show that for a general base space B which has first Chern class $c_1(B)$, that if we want to fibre an elliptic curve over this in the same fashion as Equation (4.4) we require
>
> $$O_f = 3c_1(B) \qquad \text{and} \qquad O_g = 2c_1(B).$$
>
> *Hint: Note that $c_1(B) = \sum_b Q_2^b$ where b runs over the base space indices. This follows from Equation (3.9) with $A = B$.*

Lastly we want to ask "what variety does this defining equation lie in?" i.e. which option in Equation (4.3) are we considering? To do this we recall that the weightings are related to the generating vectors by the requirement that $\sum_i Q_j^i v_i = 0$ for all j, so we have

$$3v_y + 2v_x + v_w = (0, 0, 0) = 6v_y + 4v_x + v_0 + v_1.$$

As per Equation (4.3), we have $v_x = (1, 0, 0)$, $v_y = (0, 1, 0)$ and $v_w = (-2, -3, 0)$. Then considering the options for v_0 and v_1, we see the only choices are

v_0	v_1
$(-2, -3, 1)$	$(-2, -3, -1)$
$(-4, -6, 1)$	$(0, 0, -1)$
$(0, 0, 1)$	$(-4, -6, -1)$

We then construct the toric varieties corresponding to the fans. So in total, we can form an elliptic K3 surface as a hypersurface in one of the above toric varieties via the defining equation

$$P = -y^2 + x^3 + f_8(z_0, z_1)w^4 x + g_{12}(z_0, z_1)w^6. \tag{4.5}$$

Remark 4.2.1. We should note that we could also have chosen to give w negative scaling under the $\mathbb{CP}^1$ scaling, i.e. use weight system

y	x	w	z_0	z_1	P
3	2	1	0	0	6
0	0	-2	1	1	0

This would still require $f(z_0, z_1) = f_8(z_0, z_1$ and $g(z_0, z_1) = g_{12}(z_0, z_1)$.

Calabi-Yau Check

Let's check that this does indeed correspond to a Calabi-Yau. From our weight system we have 2 linearly independent divisors H_1 and H_2, and our coordinate divisors are related to these as

$$D_y = 3H_1 + 6H_2, \qquad D_x = 2H_1 + 4H_2, \qquad D_w = H_1 \qquad \text{and} \qquad D_0 = D_1 = H_2.$$

So our ambient toric variety has Chern class

$$c(A) = (1+D_y)(1+D_x)(1+D_w)(1+D_0)(1+D_1) = (1+3H_1+6H_2)(1+2H_1+4H_2)(1+H_1)(1+H_2)^2,$$

while the defining equation has

$$c(P) = 1 + 6H_1 + 12H_2.$$

Our hypersurface then has total Chern class

$$c(X) = \frac{\prod_i (1 + H_i)}{\sum_j (1 + p_j D_j)},$$

which we can easily check has first Chern class

$$c_1(X) = (3 + 2 + 1 - 6)H_1 + (6 + 4 + 2 - 12)H_2 = 0,$$

and so it is Calabi-Yau.

Euler Characteristic

We now want to find the Euler characteristic of this space. It is a K3 surface and so we already know that the answer should be $\chi = 24$, but let's check this works out. As always, the way we do this is by finding the top form and then integrating it over X, which we shall lift to an integral over A using the normal bundle, i.e. wedging by $c_1(P)$.

As we explained before, in order to compute the Euler characteristic we need to work our the divisor intersection numbers. In this particular case, our ambient space is 3-dimensional, and so we need to work our $H_i \cdot H_j \cdot H_k$ for all $i, j, k = 1, 2$. As before, we do this by considering which cones span a smooth cone, and also which don't span a cone at all (i.e. lie in the exceptional set). For concreteness, we pick the set of vectors

v_y	v_x	v_w	v_0	v_1
0	1	-2	-4	0
1	0	-3	-6	0
0	0	0	-1	1

From here we can see that $\{v_0, v_1\}$ does *not* span a cone in Σ. Explicitly, we would have

$$\sigma_{01} = \{(-4a, -6a, b - a) \,|\, a, b \in \mathbb{R}^+\},$$

however the intersection of this with

$$\sigma_{xw} = \{(c - 2d, -3d, 0) \,|\, c, d \in \mathbb{R}^+\}$$

is given by

$$-4a = c - 2d, \qquad -6a = -3d, \qquad \text{and} \qquad b - a = 0$$

so $2b = 2a = d$ and then $c = 0$. So the intersection is $\{-4a, -6b, 0\} = 2v_w$, which is a face of σ_{xw} but it not a face of σ_{01} and so the latter cannot be a cone.

So we instantly have $D_0^2 = D_1^2 = H_2^2 = 0$. This is something we could have actually said before: H_2 corresponds to a fibre of a ruled surface (as our base space is $\mathbb{CP}^1$, and so by Proposition 3.3.11 we expect $H_2^2 = 0$.

So we are only left with H_1^3 and $H_1^2 \cdot H_2$. We now use the fact that $\{v_x, v_y, v_1\}$ are clearly a basis for $\mathbb{Z}^3$ and so correspond to a smooth space. We therefore have[3]

$$1 = D_x \cdot D_y \cdot D_1 = (3H_1 + 6H_2) \cdot (2H_1 + 4H_2) \cdot H_2 \qquad \implies \qquad 1 = 6H_1^2 \cdot H_2.$$

Finally we use that $\{v_x, v_y, v_w\}$ does not give a cone: it would correspond to the origin of $\mathbb{WCP}^2_{321}$, which we know is in the exceptional set.[4] and so

$$0 = D_x \cdot D_y \cdot D_w = (3H_1 + 6H_2)(2H_1 + 4H_2)H1$$
$$\implies 0 = H_1^3 + 4H_1^2 \cdot H_2$$
$$\implies H_1^3 = -\frac{2}{3},$$

where we have used the $H_1^2 \cdot H_2 = 1/6$ result.

So we now compute $c_2(X)$ so that we can find χ. Using the formulas above, we can easily check that we get

$$c_2(X) = 11H_1^2 + 46H_1 \cdot H_2$$

and so

$$\chi = \int_X c_2(X)$$
$$= \int_A c_1(P)c_2(X)$$
$$= \int_A (6H_1 + 12H_2)(11H_1^2 + 46H_1 \cdot H_2)$$
$$= (6 \cdot 46 + 12 \cdot 11) \int_A H_1^2 \cdot H_2 + 6 \cdot 11 \int_A H_1^3$$
$$= \frac{1}{6}(6 \cdot 46 + 12 \cdot 11) - \frac{2}{3}(6 \cdot 11)$$
$$= 46 + 22 - 44$$
$$= 24,$$

where we have tried to be reasonably explicit for clarity.

---

[3]These relations are meant to understood as being integrated over.

[4]We could have seen the $D_0 \cdot D_1 = 0$ result this way too, but it was instructive to use the above argument.

Singularity Structure

We now want to look at the singularity structure of the elliptic fibre. As we have mentioned a few times, an elliptic curve is singular when the discriminant vanishes, i.e. $\Delta = 4f^3 + 27g^2 = 0$. Well we have seen that for our elliptic K3 surface we have $f = f_8(z_0, z_1)$ and $g = g_{12}(z_0, z_1)$. So our space has $3 \cdot 8 = 2 \cdot 12 = 24$ possible singular points, i.e. there are 24 solutions to $\Delta = 0$. We now instantly see that this is exactly the Euler characteristic of our space.

Find out why.

(-2)-Curves

Before moving on to talk about elliptic 3-folds we want to make a quick comment on what are known as (-2)-curves. These are basically just sections of an elliptic K3 surface, and we shall see where this name comes from now.

Let's consider a generic curve C in an elliptic K3 surface. This is a one-dimensional sub variety, and we have

$$c_1(C) = c_1(N_C) + c_1(T_C),$$

where N_C and T_C are the normal and tangent bundles to C, respectively. This follow from our usual short exact sequence story. Now $c_1(C) = c_1(K3)|_C$, and, as a K3 surface is Calabi-Yau, we have $c_1(C) = 0$. If we then integrate the right-hand side over C, the $c_1(T_C)$ term is going to give us the Euler characteristic of the curve C, which we can write in terms of the genus of C:

$$\int_C c_1(T_C) = \chi_C = 2 - 2g_C.$$

Now we note that we can think of the self intersection of C as the zeros of a section in the normal bundle. Why? Well the normal bundle corresponds to deforming the curve (we can think of N as being little "strings" attached perpendicularly to C that we can "pull" to deform C) to a homologically equivalent curve. Any points that C intersects itself will remain invariant under this infinitesimal deformation, as they must still self intersect after. So we see that we can think of this deformation as a section in the normal bundle and the number of zeros of this section corresponds to the self intersection C^2.[5] Ah, but we have already seen/said way back in Section 2.2.2 that the zeros of a section of a line bundle is the integral of the first Chern class of that line bundle! So we have

$$\int_C c_1(N_C) = C^2$$

Putting this together we obtain

$$C^2 = 2g_C - 2.$$

So if we are considering a section in the elliptic K3 surface, which is isomorphic to $\mathbb{P}^1$ (the base space), which in turn is isomorphic to S^2, we have $g_C = 0$, and so $C^2 = -2$. This is what we refer to as a (-2)-curve. In our case we have a (-2)-curve corresponding to D_w, i.e. $w = 0$ is a section of the elliptic K3, which can be seen from Equation (4.5): if we set $w = 0$ then our defining equation has no $[z_0 : z_1]$ dependence, and so must be a section.

---

[5]This is maybe a slight abuse of notation. C is the curve, we should really talk about the self intersection of the divisor, but the idea is clear.

Now note that if we want to obtain a smooth K3 surface from our space, we need to blow up our 24 singular points. These singular points are orbifold singularities, and we have shown that the blow up of such singularities correspond to subdiving a fan, i.e. introducing a new ray. This ray has an associated toric divisor, the exceptional divisors. We also saw that this blow up procedure corresponded to including a $\mathbb{P}^1 \cong S^2$ into the space. We therefore see that exceptional divisors correspond to (-2)-curves.

We can word this in a slightly different manner: as we just said an exceptional divisor corresponds to inserting a $\mathbb{P}^1$ factor into the toric variety. These have vanishing genus, and so $\int_E c_1(T_E) = 2$, where we are using E to denote this space. If we want our blown up space to be Calabi-Yau we need to make sure the first Chern class doesn't change, i.e. remains 0. The first Chern class changes exactly as

$$c_1(N_E) + c_1(T_E),$$

and so we need to cancel this $+2$ factor, which is obtained by $E^2 = -2$.

Remark 4.2.2. It is important to note that this result is specific to K3 surfaces. This follows from the fact that a hypersurface in a K3 space is a curve, i.e. it's one-dimensional. It is precicely for this reason that we had $\int_C c_1(T_C) = \chi_C$. When then need the Calabi-Yau condition to set the left-hand-side to 0. In fact we showed back in Example 3.3.13 that F_1 (which is a 2D toric variety) corresponds to the blow of $\mathbb{CP}^2$, with $D_4 = H_1$ being the exceptional divisor. However when $n = 1$ we have $D_4^2 = -1$.

4.2.2 Over F_n: An Elliptic 3-Fold

We now want to repeat this story but for an elliptic 3-fold formed by fibering $\mathbb{WCP}^2_{321}$ over F_n. We will be a lot quicker through this calculation, and a lot of the details are left as "do I understand what's going on checks".

We have weight system

	y	x	w	z_1	z_2	z_3	z_4	P
$H_1 \to$	3	2	1	0	0	0	0	6
$H_2 \to$	$6+3n$	$4+2n$	0	1	n	1	0	$12+6n$
$H_3 \to$	6	4	0	0	1	0	1	12

and so if we want an elliptic fibration, i.e.

$$P = -y^2 + x^3 + f(z)w^4 x + g(z)w^6$$

we require $f(z)$ to have weights $(0, 8 + 4n, 8)$ and $g(z)$ to have weights $(0, 12 + 6n, 12)$, where the three entries correspond to going down the columns in the weight diagram.

> **Exercise**
>
> Using the above weight diagram, show that for $n = 0$ the only non-vanishing 4-point intersections are
>
> $$H_1^4 = 2, \qquad H_1^2 \cdot H_2 \cdot H_3 = \frac{1}{6}, \qquad H_1^3 \cdot H_2 = -\frac{1}{3} \qquad \text{and} \qquad H_1^3 \cdot H_3 = -\frac{2}{3}.$$
>
> *Remark 4.2.3.* You should get $H_2^2 = H_3^2 = 0$. Note that this should make sense in terms of ruled surfaces; think about what H_2 and H_3 represent.

In order to save doing essentially the same steps that we have already done, we do not compute the Euler characteristic of this space. We simply leave all of this as work for any readers.

As a final comment we mention that from here one can go on to discuss for which values of n is such an elliptic 3-fold smooth. Here we mean the entire 3-fold, not just the fibres as we discussed above when getting 24. The basic idea is to try factor out a z_1^m factor in f, g and Δ. There is then a table (see Table 1 of [6]) that tells you the type of singularity. We do not discuss this here as it would involve introducing the ADE singularity results, which will take up more space.

5 | Quick Summary

Let's just quickly summarise the material we have covered.

- In Chapter 1 we covered the preliminary material of orbifolds and algebraic varieties. This was mainly a list of definitions.

- We then began to study divisors in Chapter 2. We first looked at Weil divisors as formal linear combinations of irreducible hypersurfaces. We then went on to look at the subset of Weil divisors known as principal divisors. These the Weil divisors that come from a meromorphic function $f \in K(X)$. Using this definition we were able to introduce the Weil divisor class group, before moving on to discuss Cartier divisors, which are elements of a cohomology group. A key result for us was that the groups of Weil and Cartier divisors are isomorphic.

  Finally we saw the divisors are related to line bundles, clarifying why we used the terminology "hyperplane bundle" in the Complex Manifolds notes. We used this line bundle relation to talk about the first Chern class associated to a divisor. In particular we showed that the total Chern class of an ambient space is given by Equation (2.7), while the total Chern class of a defining polynomial is given by Equation (2.8). This proved very useful later when trying computing Chern classes from our toric diagrams.

- Then in Chapter 3 we started to discuss the actual toric geometry. We started with the basic definitions of toric varieties and fans/cones. We then showed how to use fans to construct toric varieties resulting in the final result Equation (3.4).

  After introducing our examples, we then introduced the weight systems and how to check if the resulting toric varieties would be compact and singular. This lead into a discussion of the blow up of orbifold singularities by subdividing a fan. We showed that this was related to the intuition of literally "blowing up" a sphere at a point.

  We then introduced toric divisors and discussed the linear relations between them and demonstrated that a toric diagram is Calabi-Yau if the sum of the toric divisors is zero. This is equivalent to saying that the generating vectors all line in the same affine hyperplane.

  Next we discussed intersection numbers and fibration structure. We showed how to use the fan diagrams to compute the intersection numbers between the toric divisors and whether the space contains a known subvariety as a subspace, in particular in relation to fibration structures. We then returned to our examples and hopefully grounded all that dense information.

After a quick discussion of how to go backwards, i.e. using toric varieties to construct fans, we introduced polytopes. These were convex hulls in our lattices. We showed how polytopes can be used to produce fans and toric varieties. This lead into the discussion of the Newton polytope and its dual. We saw that we can use a polytope to produce a basis for the group of holomorphic sections via Equation (3.17). We then specialised to the case of Calabi-Yau hypersurfaces and introduced reflexive polytopes.

- Finally in Chapter 4 we briefly discussed elliptic fibrations. The reasoning for this was two fold: firstly it allowed us to see just how easily we can produce highly complicated toric varieties and Calabi-Yau hypersurfaces using the toric geoemtery techniques; and secondly because elliptic fibrations play a key role in the study of F-theory.

  We worked through the example of an elliptic K3 surface formed as a hypersurface in $\mathbb{WCP}^2_{321}$ fibred over $\mathbb{CP}^1$. We checked that this space was indeed a Calabi-Yau and also computed its Euler characteristic. We then discussed the singularity structure of the fibres and then concluded this example by introducing (-2)-curves.

  Finally we very briefly demonstrated how this idea generalises to higher dimensional cases, by giving the weight system for an elliptic Calabi-Yau 3-fold using F_n.

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