QFT I (Path Integrals)

Course delivered in 2019 by

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Acknowledgements

These are my notes on the 2019 lecture course "QFT I" taught by Dr. Douglas Smith at Durham University as part of the Particles, Strings and Cosmology Msc. For reference, the course lasted 8 hours and was taught over the course of 4 weeks. This course is really meant more as an introduction to path integrals then a study of QFT. The module QFT II will do the actual QFT.

I have tried to correct any typos and/or mistakes I think I have noticed over the course. I have also tried to include additional information that I think supports the taught material well, which sometimes has resulted in modifying the order the material was taught. Obviously, any mistakes made because of either of these points are entirely mine and should not reflect on the taught material in any way.

I would like to extend a message of thanks to Dr. Smith for teaching this course.

If you have any comments and/or questions please feel free to contact me via the email provided on the title page.

For a list of other notes/works I have available, visit my blog site

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These notes are not endorsed by Dr. Smith or Durham University.

Richie Dadhley

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1 | Path Integrals & Feynman Kernel

This course is the study of path integrals in quantum mechanics (QM), with a particular emphasis on concepts related to quantum field theory (QFT). The first obvious questions to ask are "what are path integrals?" and "why do we care?" The first question will be answered as we go along, but allow us to answer the second now, providing motivation for the course.

1.1 Why Do We Need Path Integrals?

When we are first taught QM it is almost always taught[1] in the language of Schrödinger wavefunctions and Dirac bra-ket notation. The central equation is the Schrödinger equation

$$i\frac{d}{dt}\left|\psi\right\rangle = \hat{H}\left|\psi\right\rangle,$$

where $\hat{H}$ is the operator version of the classical Hamiltonian. As such it is clear that this approach to QM is based on the *Hamiltonian* formulation of classical mechanics.

A person who is familiar with classical field theory, will know that there is a completely different way to study classical mechanics; in terms of the *Lagrangian*. The path integral formulation approaches QM from this perspective. That is the Schrödinger equation does not appear, but instead we study principles of least action and partition functions.

Now the latter generally involves *a lot* of integral manipulation,[2] and so you would be justified in asking "why on Earth would we put ourselves through all this pain when we could just use the Hamiltonian approach to QM?" The answer to that question is the name of this course... QFT. If you are reading these notes I think it's fair to assume you have had some introduction to QFT, and it was most definitely done in the language of so-called *second quantisation*. This is the Hamiltonian approach to QFT and is obviously a very powerful tool. However, you will probably have been told about the perils of breaking manifest Lorentz invariance[3] in this approach, and then having to make checks along the way to fix your abuse. For example, you have to introduce a factor of $1/\sqrt{2E_{\vec{p}}}$ in order to get a Lorentz invariant measure, which then means you need to renormalise your states.

Of course this all works, but it would be nice if we could formulate the study of QFT in a manifestly Lorentz invariant way. This is where the path integral formulation comes in. Lagrangians are essentially *defined* to be manifestly Lorentz invariant, so if we make them

[1] If you were taught Path integral QM first, fair play.

[2] Trust me, you'll agree shortly.

[3] For clarity, by "manifest" Lorentz invariant we mean you can just look at the equations and say "yup that's Lorentz invariant."

our central objects, we're good to go. I believe this is motivation enough for you to believe that its at least worth studying. So let's get to it.

Remark 1.1.1. It is worth clarifying, it is not that the Hamiltonian doesn't appear in the path integral approach, its that we do not make it the central object. It is the Lagrangian that we are fundamentally interested in.

1.2 What Do We Want To Calculate?

For simplicity, in this course we will just focus on 1-dimensional mechanics, i.e. a particle with position $q(t)$ and momentum $p(t)$, with standard kinetic energy

$$\frac{m\dot{q}^2}{2} = \frac{p^2}{2m},$$

and potential $V(q)$. The generalisation to higher dimensions is not hard to see, but involves carrying around a bunch of sums/dot products, and as we shall see there's enough symbols as is so let's not make things worse. We therefore have a time-independent Hamiltonian

$$\hat{H} = \frac{\hat{q}^2}{2m} + \hat{V}(\hat{q}) \tag{1.1}$$

and action

$$S = \int dt\, L(q, \dot{q}), \qquad L(q, \dot{q}) = \frac{m}{2}\dot{q}^2 - V(q). \tag{1.2}$$

The basic object we will calculate using path integrals is the *Feynman kernel*

$$K(q_F, t_F; q_I, t_I) := \langle q_F | \, \hat{U}(t_F, t_I) \, | q_I \rangle \,, \tag{1.3}$$

where

$$\hat{U}(t, t_0) := \exp\left(-\frac{i}{\hbar}\hat{H}(t - t_0) \right) \tag{1.4}$$

is the *time evolution operator*. The Feynman kernel tells us the amplitude for a particle that is initially at position q_I at time t_I to evolve to position q_F at time t_F, where of course $t_F > t_I$. As such, the Feynman kernel contains all the information about the dynamics of the system, which is why we care about it so much.

Remark 1.2.1. Note that for a time-independent Hamiltonian, we have

$$\hat{U}(t, t_0) = \hat{U}(T) = \exp\left(-\frac{i}{\hbar}\hat{H}T \right),$$

where $T = t - t_0$. So the Feynman kernel becomes

$$K(q_f, t_F; q_I, t_I) = K(q_F, T; q_I, 0),$$

where $T = t_F - t_I$. We will use this from now on, unless otherwise specified.

The trick to getting the Feynman integral is to insert a complete set of states

$$\mathbb{1} = \int dq \, |q\rangle \, \langle q|$$

for both the initial and final state. Let's give examples for clarity.

Example 1.2.2. The amplitude for some initial state $|\psi\rangle$ to evolve to some final state $|\phi\rangle$ in time T is

$$\begin{aligned}
\langle\phi| \, \hat{U}(T) \, |\psi\rangle &= \int dq_I dq_F \, \langle\phi|q_F\rangle \, \langle q_F| \, \hat{U}(T) \, |q_I\rangle \, \langle q_I|\psi\rangle \\
&= \int dq_I dq_F \, \langle\phi|q_F\rangle \, K(q_F, T; q_I, 0) \, \langle q_I|\psi\rangle \\
&= \int q_I q_F \, \phi^*(q_F) K(q_F, T; q_I, 0) \psi(q_I),
\end{aligned}$$

where at the last line we've used the usual QM notation for the position space representation of a state:

$$\psi(q) := \langle q|\psi\rangle \, .$$

Example 1.2.3. Let's assume we have some state with initial wavefunction $\psi(q_I, t_I)$. We can find the expression for its evolution to some later wavefunction $\psi(q_F, t_F)$ in terms of the Feynman kernel:

$$\begin{aligned}
\psi(q_F, t_F) &:= \langle q_F|\psi(t_F)\rangle \\
&= \langle q_F| \, \hat{U}(T) \, |\psi(t_I)\rangle \\
&= \int dq_I \, \langle q_F| \, \hat{U}(T) \, |q_I\rangle \, \langle q_I|\psi(t_I)\rangle \\
&= \int dq_I K(q_F, T; q_I, 0) \psi(q_I, t_I),
\end{aligned}$$

where we have used the position space representation definition given at the end of the last example and also the definition of the time evolution operator,

$$|\psi(t_2)\rangle = \hat{U}(t_2, t_1) \, |\psi(t_1)\rangle \, ,$$

along with $T = t_F - t_I$.

To give some foresight, we will also use the Feynman kernel to calculate what are called *time-ordered correlations functions* or *Green's functions*,

$$G(t_1, t_2, ..., t_N) := \langle\Omega| \, \mathcal{T}\big\{\hat{q}(t_1)\hat{q}(t_2)...\hat{q}(t_N)\big\} \, |\Omega\rangle \, ,$$

where $|\Omega\rangle$ is the ground state of our system, and $\mathcal{T}$ is the time ordering operator, which essentially orders the Heisenberg[4] picture operators in increasing time, e.g.

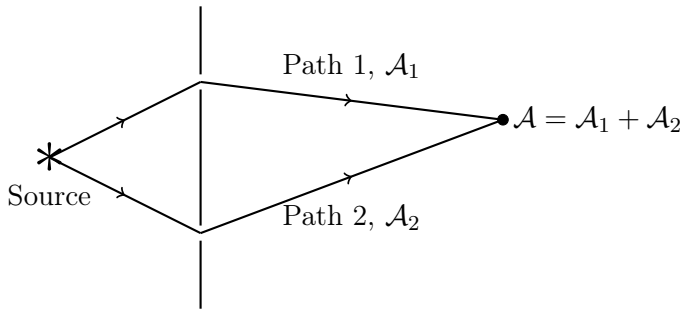
$$\mathcal{T}\big\{\hat{\mathcal{O}}_1(t_1)\hat{\mathcal{O}}_2(t_2)\big\} = \begin{cases} \hat{\mathcal{O}}_1(t_1)\hat{\mathcal{O}}_2(t_2) & \text{if } t_1 > t_2 \\ \hat{\mathcal{O}}_2(t_2)\hat{\mathcal{O}}_1(t_1) & \text{if } t_2 > t_1 \end{cases} .$$

Notation. From now on I am going to drop the hats on operators in order to save typing. It should be reasonably clear from the context, and whenever potential confusion might arise I shall try be explicit.

[4] As otherwise the operators are time independent

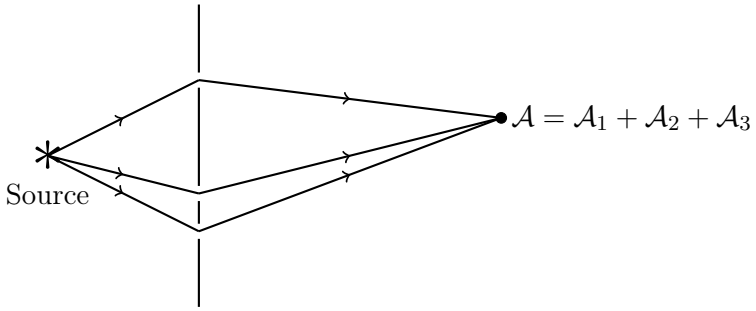
1.3 Motivating The Path Integral Approach

So what is a path integral? Well as the name suggests, it is an integral over paths, but what do we mean by this and what's it got to do with the Feynman kernel? Well let's consider the double slit experiment. The interference pattern this generates can be explained in terms of waves, or we can explain it in terms of particles, provided we put a phase on each particle path. That is, the total amplitude to go from the source to a particular point is given by the sum of the amplitudes of the two separate paths, as indicated in the figure below.



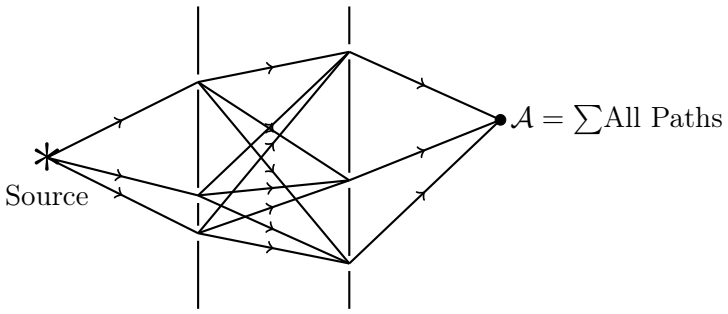
1.3.1 Zee's "Wise Guy"

The story then goes as follows:[5] a teacher had just finished explaining the above idea to his class when a student, going by the name of Feynman, chimed in and said "Excuse me sir, but what happens if I put another slit in my screen?" The lecturer then replied "We just add another contribution" while drawing the following diagram



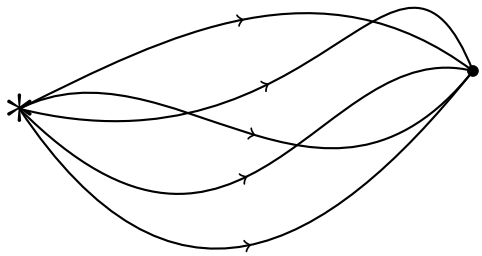
The professor then went back to what he was about to say, when Feynman chimed in again: "What if I now poke another hole in my screen? Then another, and another after that?" The professor at this point had had enough and simply said "Ok wise guy, I think it's clear to everyone that you simply sum over all the different path amplitudes. Now where was I..." But before he could continue, yup that's right Feynman chimed in again. "What about if I include another screen?" Slightly irritated, but bound by his profession to be polite, the professor drew the following diagram

[5] I'm paraphrasing Zee's book here, as I love this description.



Now before Feynman could chime in again the professor said "And before you ask, if I add another screen, I just do the same thing and sum over all paths again." However Feynman had one more, very important, chime in. "Ok Sir," he said, ignoring the frustrated look on the professor's face, "what if I include an infinite number of screens and on each screen I poke an infinite number of holes, so that in total there are no screens and no holes?" The professor just laughed and continued with the rest of his course.

But as Zee points on, this last comment has a very important result. If we want to find the amplitude for a particle to propagate from some source and arrive at a detector, we must consider *every single path* connecting the two, and sum over all their amplitudes.



However there is obviously an infinite number of paths connecting the two and so the result is not a sum, but an integral. And thus the path integral is born.

Remark 1.3.1. On a technical note, we require the paths we integrate over to be continuous.[6] This is a reasonable thing to assume, as a discontinuous line would correspond to a particle 'teleporting' while a non-differentiable one would mean its velocity was discontinuous, which is also a problem.

1.3.2 Relation To Feynman Kernel

Ok so we've answered the first question about what a path integral is, now we just need to see how it's related to the Feynman kernel. Well in the case of 1-dimensional QM the analogy

[6] On an even more technical note, it is important that the derivatives of these paths are not continuous, i.e. if we denote the paths by γ then $\gamma \in C^0(\mathcal{M})$ but $\gamma \notin C^1(\mathcal{M})$. We direct the reader interested in why this is the case to Section 3.2 of Dr. Skinner's QFT II notes. Many thanks to Rudolfs for pointing this out to me!

is that each screen corresponds to a fixed time and each slit a position at that time. So the path between two slits is telling us the amplitude to go from one position at one time to another position at a later time. Well this is just the Feynman kernel. So in the 'limit' that we remove all the screens, our Feynman kernel is an integral over all the different paths through spacetime connecting the initial position at initial time to the final position at the final time.

Remark 1.3.2. Note to go to a higher dimensional case is easy, simply replace the slits with a d-dimensional sheet with holes poked in it.

1.4 Hamiltonian Phase Space

Let's calculate something now. It might not be obvious at first what we're calculating, but it will become clear next lecture. In order to do this, first you need to complete the following exercise.

Exercise

Given that in the position space representation, a wavefunction for a momentum eigenstate with momentum p is

$$\psi_p(q) := \langle q|p\rangle\,, \qquad -i\hbar\frac{d}{dt}\psi_p(q) = p\psi_p(q),$$

show that

$$\langle p|q\rangle = (\langle q|p\rangle)^* = \frac{1}{\sqrt{2\pi\hbar}}\exp\left(-\frac{i}{\hbar}pq\right). \tag{1.5}$$

Hint: Use the differential equation above to guess the form of $\psi_p(q)$ and then impose a normalisation condition.

So what we want to find is

$$\langle p_F|\,U(T)\,|q_I\rangle = \langle p_F|\,e^{-\frac{i}{\hbar}HT}\,|q_I\rangle\,.$$

Note this is *not* the Feynman kernel as we have p_F for the left hand state. With the above intuition in mind, let's break the the time interval up into n equal intervals with

$$t_j = t_I + j\epsilon, \qquad \text{with} \qquad \epsilon = \frac{T}{n}.$$

Note that this gives us $t_0 = t_I$ and $t_n = t_F$, which is obviously what we want. We shall therefore relabel $p_F = p_n$ and $q_I = q_0$. Using the ϵ definition we can rewrite the our expression as

$$\langle p_n|\,e^{(-\frac{i}{\hbar}H\epsilon)^n}\,|q_0\rangle = \langle p_n|\,\underbrace{e^{-\frac{i}{\hbar}H\epsilon}e^{-\frac{i}{\hbar}H\epsilon}...e^{-\frac{i}{\hbar}H\epsilon}}_{n\text{-times}}\,|q_0\rangle\,.$$

Now we employ our trick of inserting the identity as a complete set of states. In fact we insert

$$\mathbb{1} = \mathbb{1}^2 = \int dq\,|q\rangle\,\langle q|\int dp\,|p\rangle\,\langle p| = \int\frac{dqdp}{\sqrt{2\pi\hbar}}e^{\frac{i}{\hbar}pq}\,|q\rangle\,\langle p|\,,$$

where we have used Equation (1.5). We do this in-between every exponential term. This gives us[7]

$$\langle p_n|\, U(T)\,|q_0\rangle = \int \left[\prod_{i=1}^{n-1} \frac{dq_i dp_i}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar}p_i q_i}\right] \langle p_n|\, e^{-\frac{i}{\hbar}H\epsilon}\,|q_{n-1}\rangle \langle p_{n-1}|\, e^{-\frac{i}{\hbar}H\epsilon}\,|q_{n-2}\rangle \ldots \langle p_1|\, e^{-\frac{i}{\hbar}H\epsilon}\,|q_0\rangle\,.$$

We now take the limit $n \to \infty$ (an infinite number of screens), this gives $\epsilon \to 0$, and so we can Taylor expand the exponentials:

$$e^{-\frac{i}{\hbar}H\epsilon} = \mathbb{1} - \frac{i}{\hbar}H\epsilon + \mathcal{O}\left(\frac{1}{n^2}\right).$$

Then using Equation (1.1), we have[8]

$$\begin{aligned}\langle p_i|\, \hat{H}\,|q_{i-1}\rangle &= \frac{1}{2m}\langle p_i|\, \hat{q}^2\,|q_{i-1}\rangle + \langle p_i|\, \hat{V}(\hat{q})\,|q_{i-1}\rangle \\ &= \left(\frac{q_{i-1}^2}{2m} + V(q_{i-1})\right)\langle p_i|q_{i-1}\rangle \\ &= \frac{1}{\sqrt{2\pi\hbar}} H(p_i, q_{i-1}) e^{-\frac{i}{\hbar}p_i q_{i-1}},\end{aligned}$$

which gives us

$$\langle p_i|\, e^{-\frac{i}{\hbar}\hat{H}\epsilon}\,|q_{i-1}\rangle = \frac{1}{\sqrt{2\pi\hbar}} \exp\left(-\frac{i}{\hbar}\big(p_i q_{i-1} + \epsilon H(p_i, q_{i-1})\big)\right) + \mathcal{O}\left(\frac{1}{n^2}\right).$$

Putting this into our expression above, we get

$$\begin{aligned}\langle p_n|\, U(T)\,|q_0\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int \left[\prod_{i=1}^{n-1} \frac{dq_i dp_i}{2\pi\hbar}\right] \exp\Bigg\{\frac{i}{\hbar}\Bigg[\sum_{i=1}^{n-1}\left(\frac{p_i(q_i - q_{i-1})}{\epsilon} - H(p_i, q_{i-1})\right)\epsilon \\ - p_n q_{n-1} - H(p_n, q_{n-1})\epsilon\Bigg]\Bigg\} + \mathcal{O}\left(\frac{1}{n^2}\right).\end{aligned}$$

We then take the limit $n \to \infty$ and define the path integral measure

$$\int DpDq := \lim_{n\to\infty} \int \left[\prod_{i=1}^{n-1} \frac{dq_i}{\sqrt{2\pi\hbar}} \frac{dp_i}{\sqrt{2\pi\hbar}}\right], \tag{1.6}$$

and then argue that $q_{n-1} = q_n$ in this limit, and also use

$$\lim_{n\to\infty} \frac{q_i - q_{i-1}}{\epsilon} = \dot{q}(t_i)$$

i.e. the velocity of the particle,[9] to finally give us

[7] And so begins the painful job of following terms around...
[8] Here I reinsert the hats for clarity
[9] Note this is well defined given Remark 1.3.1

$$\langle p_n|\, U(T)\, |q_0\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int DpDq \exp\left[\frac{i}{\hbar}\left(\int_{t_I}^{t_F} dt\Big(p(t)\dot{q}(t) - H(p,q)\Big) - p(t_F)q(t_F)\right)\right]. \tag{1.7}$$

1.5 Wick Rotation

As promised above, next lecture we will see why we have bothered to derive Equation (1.7), but in order to do that we will need to use a very popular trick, the *Wick rotation*, and we conclude this lecture with this.

As we will see, we are going to take the limit $t \to \infty$ next lecture, and so we need to work out if such a thing is well defined for our time-evolution operator. That is we want to ask what happens to[10]

$$U(t)\,|\psi\rangle = \int dE e^{-\frac{i}{\hbar}Ht}\,|E\rangle\,\langle E|\psi\rangle = \int dE e^{-\frac{i}{\hbar}Et}\,|E\rangle\,\langle E|\psi\rangle \tag{1.8}$$

in the limit $t \to \infty$. Well consider the analytic continuation of t to the complex plane:

$$t \to e^{-i\theta}t, \qquad \theta \in [0, \pi/2]. \tag{1.9}$$

This is what we call a Wick rotation, and we note that $\theta = 0$ changes nothing, but $\theta = \pi/2$ takes our exponential in Equation (1.8) from being complex to being real. People often refer to this as "going to Euclidean time", for obvious reasons. Ok so what happens no? Well our expression becomes (setting $\hbar = 1$ for this calculation)

$$U(t)\,|\psi\rangle = \int dE e^{-i\cos\theta Et} e^{-\sin\theta Et}\,|E\rangle\,\langle E|\psi\rangle\,.$$

Now if we assume we have a unique ground state $|\Omega\rangle$, scaled such that $\langle\Omega|\psi\rangle \neq 0$ and that we have a mass gap,[11] then the second exponential in the above expression suppresses all other energy states in our limit. That is,

$$\lim_{t\to\infty} U(t)\,|\psi\rangle = e^{-i\cos\theta E_\Omega t} e^{-\sin\theta E_\Omega t}\,|\Omega\rangle\,\langle\Omega|\psi\rangle\,. \tag{1.10}$$

Exercise

Convince yourself that similarly to Equation (1.10), we have

$$\lim_{t\to\infty} \langle\psi|\, U(t) = e^{-i\cos\theta E_\Omega t} e^{-\sin\theta E_\Omega t}\,\langle\Omega|\,\langle\psi|\Omega\rangle\,.$$

[10] Note we've inserted $\mathbb{1} = \int dE\,|E\rangle\,\langle E|$.

[11] That is the next energy level is strictly greater then E_Ω

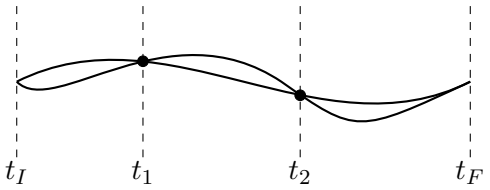
2 | Green's Functions

2.1 N-Point Green's Function

Last lecture we derived an expression for $\langle p_n| U(T) |q_0\rangle$ and said we would use it for something. Let's now consider the same calculation, but now we insert a bunch of q operators at times $t_1 < t_2 < ... < t_N$ in between the different $U(t)$s. That is consider

$$\langle p_F| U(t_F - t_N) q\, U(t_N - t_{N-1}) q ... q\, U(t_1 - t_I) |q_I\rangle = \langle p_F| U(t_F) \mathcal{T}\big(q(t_N)...q(t_1)\big) U(t_I) |q_I\rangle .$$

This equation above says "start at some initial position and time, $|q_I\rangle$, then evolve to t_1, $U(t_1 - t_I)$, then insert a q operator at t_1, then evolve to t_2, $U(t_2 - t_1)$, insert a q, etc until you get to t_N and then finally evolve to t_F and arrive at the final state." For clarity, what we're doing is considering all the paths from initial state $|q_I\rangle$ to final state $|p_F\rangle$ and fixing N points along the way by the values $q(t_i)$. That is, every path not only has to start and end at the same place, but they must all meet at N points in between. For example, if $N = 2$ we have two fixed points, and all paths must meet there, as indicated with the two paths below. The dashed lines are our equal time sheets, and the two dots represent the insertion of an operator q.



The calculation will follow through exactly as above, but now the q operators will act on the inserted states $|q_i\rangle$. This gives us

$$\frac{1}{\sqrt{2\pi\hbar}} \int DpDq\, q(t_N) q(t_{N-1}) ... q(t_1) \exp\left[\frac{i}{\hbar}\Big(\int_{t_I}^{t_F} dt\Big(p(t)\dot{q}(t) - H(p,q)\Big) - p(t_F)q(t_F)\Big)\right],$$

where the $q(t_i)$s here are the eigenvalues.

Remark 2.1.1. Note the path integral automatically gives us the time-ordering. That is, even if we switch the $q(t_i)$s around in the final expression (which we can do because they're just numbers) we can arrive back at the time ordered expression. This is just because we define our evolution operator to only go forward in time, and so the only way we can order our time evolution operators is so that $q(t_N)$ appears furthest to the left and $q(t_1)$ furthest to the right.

We now use the result from the end of last lecture, to claim that in the limit $t_I \to -\infty$ and $t_F \to \infty$, the above simply becomes the so-called *N-point Green's function*:

$$G(t_1, ..., t_N) := \langle \Omega | \, \mathcal{T}\big(q(t_1)...q(t)N)\big) \, | \Omega \rangle \, .$$

This just comes from taking the limit using the right hand side of the first equation in this lecture. So we get an explicit expression for it as

$$G(t_1, ..., t_N) = \widetilde{\mathcal{N}} \int DpDq \, q(t_N)q(t_{N-1})...q(t_1) \exp\left[\frac{i}{\hbar}\int_{-\infty e^{-i\theta}}^{\infty e^{-i\theta}} dt \, \big(p(t)\dot{q}(t) - H(p,q)\big)\right],$$

for some normalisation constant $\widetilde{\mathcal{N}}$, which contains the $p(t_F)q(t_F)$ term in the previous exponential. Note the limits on the integrals contain factors of $e^{-i\theta}$ to remind us that we're doing the Wick rotated integral. To save notation, we shall drop the integral limits below, but it's important to note that we have to take this limit to define the N-point Green's function.

Remark 2.1.2. With that last comment in mind, it is common to take $\theta = \pi/2$ so that $e^{-i\theta} = -i$ and then define $\tau = it$ so that we can write the integral as[1]

$$i\int_{i\infty}^{-i\infty} dt = \int_{-\infty}^{\infty} d\tau$$

As mentioned last lecture, τ is referred to as *Euclidean time*. Note in Euclidean time $\dot{q} = i\dot{q}$, and so

$$L(q, \dot{q}) \to -\frac{m}{2}\dot{q}^2 - V(q),$$

and so we define the *Euclidean Action*

$$S_E[q] := \int d\tau \left(\frac{m}{2}\dot{q}^2 + V(q)\right)$$

This allows us to see the exponential suppression easier as we get (setting $\hbar = 1$ for a moment).

$$e^{iS} \to e^{-S_E}.$$

2.1.1 1-Dimensional QM

Let's evaluate this for our 1-dimensional QM system. We have

$$\begin{aligned} p\dot{q} - H &= p\dot{q} - \frac{p^2}{2m} - V(q) \\ &= -\frac{1}{2m}(p - m\dot{q})^2 + \frac{m}{2}\dot{q}^2 - V(q) \\ &= -\frac{1}{2m}(p - m\dot{q})^2 + L(q, \dot{q}), \end{aligned}$$

which allows us to split the Dp and Dq integral, giving us

$$G(t_1, ..., t_N) = \widetilde{N} \int Dp \exp\left(-\frac{i}{2m\hbar}\int dt \, (p - m\dot{q})^2\right) \int Dq \, q(t_N)...q(t_1) \exp\left[\frac{i}{\hbar}\int dt \, L(q, \dot{q})\right].$$

[1] Bonus exercise, check this.

Claim 2.1.3. The integral

$$\int Dp \exp\left(-\frac{i}{2m\hbar} \int dt\, (p - m\dot{q})^2 \right)$$

is well defined.

Proof. Given Remark 2.1.2, at first site this claim doesn't seem true, as going to Euclidean time just removes the i and puts a minus sign in, which in this case would give us a positive exponential. However we have to remember that $\dot{q}$ involves a time derivative and so is also effected. We therefore prove the claim by considering discreting time. The integral then becomes a product of integrals of the form

$$\int d\widetilde{p} \exp\left[-\frac{i\epsilon}{2m\hbar} \widetilde{p}^2 \right],$$

where we have defined

$$\widetilde{p} := (p - m\dot{q}),$$

and used the fact that p and $\widetilde{p}$ are linearly related to change the measure to $d\widetilde{p}$. Under the Wick rotation, this becomes

$$\int d\widetilde{p} \exp\left[-\frac{\epsilon}{2m\hbar} \big(\sin\theta + i\cos\theta \big) \widetilde{p}^2 \right] = \sqrt{\frac{2\pi\hbar m}{i\epsilon(\cos\theta - i\sin\theta)}},$$

where the right-hand side comes from Gaussian integral formula which states

$$\int dx e^{-ax^2} = \sqrt{\frac{\pi}{a}} \qquad \text{if} \qquad \operatorname{Re} a > 0,$$

which we have as $\theta \in [0, \pi/2]$. If we then set $\theta = 0$, we get

$$\int Dp \exp\left(-\frac{i}{2m\hbar} \int dt (p - m\dot{q})^2 \right) = \lim_{n\to\infty} \left(\frac{-im}{\epsilon} \right)^{\frac{n-1}{2}}, \tag{2.1}$$

where have used the definition of Dp, Equation (1.6). □

So if we absorb this result into our normalisation constant, and use the definition of the action, we get

$$G(t_1, ..., t_N) = \mathcal{N} \int Dq\, q(t_N)...q(t_1) \exp\left(\frac{i}{\hbar} S[q] \right).$$

We can find the normalisation constant by consider the 0-point Green's function, which is just the vacuum inner product $\langle \Omega | \Omega \rangle = 1$, as we define it to be normalised. Therefore we conclude

$$\mathcal{N} = \frac{1}{\int Dq\, \exp\left(\frac{i}{\hbar} S[q] \right)},$$

finally giving us the result

$$G(t_1,...,t_N) = \frac{\int Dq\, q(t_N)...q(t_1) \exp\left(\frac{i}{\hbar}S[q]\right)}{\int Dq\, \exp\left(\frac{i}{\hbar}S[q]\right)}. \tag{2.2}$$

Remark 2.1.4. Note that $G(t_1,...,t_N)$ is totally symmetric in all its entries, as the $q(t_i)$s are just numbers. Recall Remark 2.1.1 tells us we can do this without altering the time ordering we need.

In Euclidean time this can be written

$$G_E(t_1,...,t_N) = \frac{\int Dq\, q(t_N)...q(t_1) \exp\left(-\frac{1}{\hbar}S_E[q]\right)}{\int Dq\, \exp\left(-\frac{1}{\hbar}S_E[q]\right)}$$

2.2 Partition Function

2.2.1 Functional Derivative

Recall that a functional is an object that maps functions to numbers. The action is an example, as it maps the functions $q(t)$ to some number. Recall the derivative of a function is given by

$$\frac{df}{dx} := \lim_{\epsilon\to 0} \frac{f(x+\epsilon) - f(x)}{\epsilon}.$$

We can use this to motivate the definition of the derivative of a functional.

Definition. [Functional Derivative] Given a functional $F[J]$, we define the *functional derivative* of F as

$$\frac{\delta F[J]}{\delta J(s)} := \lim_{\epsilon\to 0} \frac{F[J(t) + \epsilon\delta(t-s)] - F[J]}{\epsilon}. \tag{2.3}$$

We are often only interested in the variation $\delta F[J]$, which is given by

$$\delta F[J] = \int ds\, \delta J(s) \frac{\delta F[J]}{\delta J(s)}. \tag{2.4}$$

We can also find the variation by simply calculating

$$F[J + \delta J] - F[J]$$

to order δJ.

Corollary 2.2.1. *It follows immediately from the definition above that for the functional*

$$F[J] = J(t), \qquad \forall J$$

where t is some fixed value in the argument of J that

$$\frac{\delta J(t)}{\delta J(s)} := \frac{\delta F[J]}{\delta J(s)} = \delta(t-s),$$

where the left-hand expression is defined for notational simplicity.

Remark 2.2.2. We can treat $\frac{\delta}{\delta J(s)}$ sort of like a regular partial derivative, and use things like the product rule. However we should be careful when doing this as it doesn't always work.

Example 2.2.3. Let

$$F[J] = \int dt \phi(t) J(t),$$

then

$$\begin{aligned}\frac{\delta F[J]}{\delta J(s)} &= \lim_{\epsilon \to 0} \frac{\int dt \phi(t)\big(J(t) + \epsilon \delta(t-s)\big) - \int dt \phi(t) J(t)}{\epsilon} \\ &= \phi(s)\end{aligned}$$

We get the same result using the partial derivative approach:

$$\begin{aligned}\frac{\delta F[J]}{\delta J(s)} &= \int dt \left(\frac{\delta \phi(t)}{\delta J(s)} J(t) + \phi(t) \frac{\delta J(t)}{\delta J(s)} \right) \\ &= \int dt \phi(t) \delta(t-s) \\ &= \phi(s).\end{aligned}$$

This gives us

$$\delta F[J] = \int ds \phi(s) \delta J(s).$$

2.2.2 Partition Function

We can take the sum over all the N-point Green's functions and get what is known as a *generating functional*. For the specific case of the sum of N-point Green's functions, we call the resulting generating functional the *partition function*.

$$Z[J] := 1 + \sum_{N=1}^{\infty} \frac{1}{N!} \int ds_1 ... ds_N G(s_1, ..., s_N) J(s_1) ... J(s_N). \tag{2.5}$$

The partition function is incredible useful as it allows us to calculate *any* N-point Green's function by taking N functional derivatives and then setting $J = 0$. That is

$$G(t_1, .., t_N) = \left. \frac{\delta^N Z[J]}{\delta J(t_1) ... \delta J(t_N)} \right|_{J=0}. \tag{2.6}$$

Example 2.2.4. Let's do it for $N = 2$. The derivatives will kill the first two terms in the expansion, and then anything with more than two Js will vanish when we set $J = 0$. So we

are left with

$$\begin{aligned}\frac{\delta^2 Z[J]}{\delta J(t_1)\delta J(t_2)}\bigg|_{J=0} &= \frac{1}{2}\int ds_1 ds_2 G(s_1,s_2)\frac{\delta}{\delta J(s_1)}\big(\delta(s_1-t_2)J(s_2)+J(s_1)\delta(s_2-t_2)\big)\\ &= \frac{1}{2}\int ds_1 ds_2 G(s_1,s_2)\big(\delta(s_1-t_2)\delta(s_2-t_2)+\delta(s_2-t_2)\delta(s_1-t_2)\big)\\ &= \frac{1}{2}\big(G(t_2,t_1)+G(t_1,t_2)\big)\\ &= G(t_1,t_2),\end{aligned}$$

where the last line has used the symmetric property of the N-point Green's functions.

2.2.3 Path Integral Form

We can use the path integral expression for the N-point Green's function, Equation (2.2), to write the partition function as a path integral. We have

$$\begin{aligned}\frac{1}{\mathcal{N}}Z[J] &= \int Dq\left(e^{\frac{i}{\hbar}S[q]}+\sum_{N=1}^{\infty}\frac{1}{N!}\int ds_1...ds_N J(s_1)...J(s_N)q(s_1)...q(s_N)\,e^{\frac{i}{\hbar}S[q]}\right)\\ &= \int Dq\sum_{N=0}^{\infty}\frac{1}{N!}\left(\int ds\, q(s)J(s)\right)^N e^{\frac{i}{\hbar}S[q]}\\ &= \int Dq\exp\left(\frac{i}{\hbar}S[q]+\int ds\, q(s)J(s)\right),\end{aligned}$$

so that

$$Z[J] = \mathcal{N}\int Dq\exp\left(\frac{i}{\hbar}S[q]+\int dt\, q(t)J(t)\right) \tag{2.7}$$

Remark 2.2.5. Equation (2.7) is often defined using another convention given by sending

$$\frac{\delta}{\delta J(t)} \to -i\hbar\frac{\delta}{\delta J(t)},$$

so that we get

$$Z[J] = \mathcal{N}\int Dq\exp\left[\frac{i}{\hbar}\left(S[q]+\int dt\, q(t)J(t)\right)\right].$$

We can also write it as a Euclidean path integral. If we use the convention Equation (2.7), we redefine $J(t)\to iJ(t)$ to give

$$Z[J] = \mathcal{N}\int Dq\exp\left(-\frac{1}{\hbar}S_E[q]+\int dt\, q(t)J(t)\right).$$

2.3 Back To The Feynman Kernel

Ok now let's return to the Feynman kernel and express it as a path integral. Recall that it's defined as

$$K(q_F, T; q_I, 0) := \langle q_F | \, U(T) \, | q_I \rangle \, .$$

We can insert the complete set of states

$$\mathbb{1} = \int dp_F \, |p_F\rangle \, \langle p_F|$$

to give

$$\begin{aligned} K(q_F, T; q_I, 0) &= \int dp_F \, \langle q_F | p_F \rangle \, \langle p_F | \, U(T) \, | q_I \rangle \\ &= \lim_{n\to\infty} \left(\frac{-im}{\epsilon} \right)^{\frac{n-1}{2}} \int dp_F \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar} p_F q_F} \frac{1}{\sqrt{2\pi\hbar}} \int Dq \exp\left(\frac{i}{\hbar} S[q] - p_F q_{n-1} \right), \end{aligned}$$

where we have used Equations (1.5), (1.7) and (2.1), with the latter manipulated as we did for the Green's function to get the action in the exponential. Note that we haven't taken the limit on the q_{n-1} in the exponential yet, this is because we can now combine it with the first exponential and use

$$\int dp_F \exp\left(\frac{i}{\hbar} p_F (q_F - q_{n-1}) \right) = (2\pi\hbar)\delta(q_F - q_{n-1}),$$

to give (putting the definition of Dq, Equation (1.6), in)

$$\begin{aligned} K(q_F, T; q_I, 0) &= \lim_{n\to\infty} \left(\frac{-im}{\epsilon} \right)^{\frac{n-1}{2}} \int \left[\prod_{i=1}^{n-1} \frac{dq_i}{\sqrt{2\pi\hbar}} \right] \delta(q_F - q_{n-1}) \exp\left(\frac{i}{\hbar} S[q] \right) \\ &= \lim_{n\to\infty} \sqrt{\frac{-im}{2\pi\hbar\epsilon}} \int \left[\prod_{i=1}^{n-2} \sqrt{\frac{-im}{2\pi\hbar\epsilon}} dq_i \right] \exp\left(\frac{i}{\hbar} S[q] \right) \Big|_{q_{n-1}=q_F}, \end{aligned} \tag{2.8}$$

where on the second line we have done some rearrangement (notice the upper limit on the product changes because of delta function). So if we redefine Dq appropriately, we get

$$K(q_F, T; q_I, 0) = \int Dq \exp\left(\frac{i}{\hbar} S[q] \right) \tag{2.9}$$

3 | Classical Limit & Schwinger-Dyson Equation

3.1 Classical Limit

Last lecture we showed that in our path integral every path contributes a factor of $e^{\frac{i}{\hbar}S[q]}$, all with magnitude 1. This was a result in the quantum theory, and as always it is instructive to check that in some limit we get the classical theory back, i.e. only paths that satisfy the Euler-Lagrange equations. We shall do just that now.

When we observe a path, there is always some uncertainty about the path, that is we cannot distinguish between $q(t)$ and $q(t)+\eta(t)$, where $\eta(t)$ is some small deviation. Now the action for this latter path is given by

$$\begin{aligned} S[q+\eta] &= \int_{t_I}^{t_F} dt\, L(q+\eta, \dot{q}+\dot{\eta}) \\ &= \int_{t_I}^{t_F} dt \left(L(q,\dot{q}) + \eta(t)\frac{\partial L}{\partial q}(q,\dot{q}) + \dot{\eta}(t)\frac{\partial L}{\partial \dot{q}}(q,\dot{q}) + \mathcal{O}(\eta^2) \right) \\ &= S[q] + \int_{t_I}^{t_F} dt\, \eta(t) \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\right] \right) + \mathcal{O}(\eta^2), \end{aligned}$$

where we've used the standard tool of integrating by parts and saying that $\eta(t_I) = 0 = \eta(t_F)$, and where on the last line we have suppressed the arguments of L for notational reasons. So we see the deviated path contributes a phase factor of

$$\int_{t_I}^{t_F} dt\, \eta(t) \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\right] \right) + \mathcal{O}(\eta^2).$$

So what does this mean for the classical limit? Well we need to sum over all the different $\eta(t)$ variations, and if $q(t)$ is not the classical path $q_{cl}(t)$, i.e. they don't satisfy the Euler-Lagrange equations, we get a wide range of contributions to the phase. Overall these will approximately destructively interfere and so will not contribute too drastically to the result. Now if $q(t) = q_{cl}(t)$ then the bracket term vanishes and the contributions to the phase are $\mathcal{O}(\eta^2)$, and so the deviation range is decreased. We claim that this results in a significant reduction in the destructive interference and contributes to constructive interference, and so this gives the biggest contribution. In other words, paths that aren't the classical path cancel whereas the ones near the classical path add, so we get the classical result.

Remark 3.1.1. We can also see this result by doing our Wick rotation to the Euclidean picture. Here we get contributions of the form

$$\exp\bigg(-\frac{1}{\hbar}\big(S_E[q]-S_E[q_{cl}]\big)\bigg),$$

and so we see straight away that non-classical paths are exponentially suppressed.

3.1.1 Variations Around The Classical Path

What happens to our path integral when we take a variation around the classical path? We've just seen how the action changes, and showed that if we take the variation around the classical path we get

$$S[q+\eta] = S[q_{cl}] + \mathcal{O}(\eta^2).$$

What about the integral measure Dq? Well to this the change $q = q_{cl} + \eta$ is just a change of variables, and we take q_{cl} to be a 'constant' so we just get

$$Dq \to D\eta.$$

So the path integral is

$$\int Dq e^{\frac{i}{\hbar}S[q]} = e^{\frac{i}{\hbar}S[q_{cl}]}\int D\eta\, e^{\frac{i}{\hbar}\mathcal{O}(\eta^2)}.$$

Now we note that if we consider an action that is at most quadratic in q, then our order η^2 terms are independent of q_{cl} (we would need at least a cubic q for that), and so for these cases we get

$$\int Dq e^{\frac{i}{\hbar}S[q]} = e^{\frac{i}{\hbar}S[q_{cl}]}\int D\eta\, e^{\frac{i}{\hbar}S[\eta]}.$$

We then simplify notation by using $t_I = 0$, $t_F = T$ which gives

$$\eta(0) = 0 = \eta(T) \qquad \Longrightarrow \qquad \eta(t) = \sum_{i=1}^{\infty} a_\lambda \sin\bigg(\frac{\lambda\pi t}{T}\bigg), \tag{3.1}$$

then if we discretise the time over n intermediate paths q_j, this is equivalent (up to a normalisation factor) to taking $\eta(t)$ as above and integrating over the a_λ. We shall return to this soon when consdiering the harmonic oscillator.

3.2 Calculating The PI Exactly

As anyone familiar with QFT knows, it is only in very special cases that we can obtain exact results and in all other cases we tend to turn to perturbation theory. We will address this next lecture, but for this lecture we will present particular examples of when we can obtain an exact result for the path integral. It is worth emphasising again that this is *not* something you can do in general, but is a particular property of the systems we're studying.

3.2.1 Free Particle

Of course the most simple system we can study is the free particle. This corresponds to setting $V(q) = 0$, and so our Feynman kernel (in the discrete time formalism, i.e. using Equation (2.8)) becomes

$$K(q_F, T; q_I, 0) = \lim_{n\to\infty} \sqrt{\frac{-im}{2\pi\hbar\epsilon}} \int \left[\prod_{i=1}^{n-2} \sqrt{\frac{-im}{2\pi\hbar\epsilon}} dq_i\right] \exp\left(\frac{im}{2\hbar\epsilon} \sum_{j=1}^{n-1} (q_j - q_{j-1})^2\right), \tag{3.2}$$

where we also impose $q_0 = q_I$ and $q_{n-1} = q_F$. We now notice that these are essentially a (huge) collection of Gaussian integrals, and so we can solve it. However, we need to be a bit clever because each q_j appears twice, i.e. once as q_j and once as $q_{(j+1)-1}$. We therefore want some kind of inductive proof of a result.

Claim 3.2.1. We can rewrite Equation (3.2) as

$$K(q_F, T; q_I, 0) = \lim_{n\to\infty} \sqrt{\frac{-im}{2\pi\hbar\epsilon}} \left[\prod_{j=1}^{n-2} \sqrt{\frac{j}{j+1}}\right] \exp\left(\frac{im}{2\hbar\epsilon} \frac{1}{n-1} (q_F - q_I)^2\right)$$

Proof. First we note that the we can do the dq_j integral without effecting any of the $k > j$ results but it will effect the $\ell < j$ ones. So we start at $j = 1$ and work upwards. The proof then follows by noticing that we can write the $j = 1$ term as (note $j_0 = j_I$).

$$\mathcal{I}_1 = \sqrt{\frac{-im}{2\pi\hbar\epsilon}} \int dq_j \exp\left[\frac{im}{2\hbar\epsilon}\left((q_{j+1} - q_j)^2 + \frac{1}{j}(q_j - q_I)^2\right)\right].$$

Let's see what this evaluates to. We start by expanding

$$(q_{j+1} - q_j)^2 + \frac{1}{j}(q_j - q_I)^2 = \left(\frac{j+1}{j}\right) q_j^2 - 2\left(q_{j+1} + \frac{1}{j} q_I\right) q_j + q_{j+1}^2 + \frac{1}{j} q_I^2.$$

Now we use a Wick rotation $T \to -iT$, and hence $\epsilon = T/n \to -i\epsilon$, so our integral becomes

$$\mathcal{I}_1^E = \sqrt{\frac{m}{2\pi\hbar\epsilon}} \int dq_j \exp\left(-aq_j^2 + bq_j + c\right)$$

where

$$a = \frac{m}{2\hbar\epsilon}\left(\frac{j+1}{j}\right), \qquad b = \frac{m}{\hbar\epsilon}\left(q_{j+1} + \frac{1}{j} q_I\right), \quad \text{and} \quad c = -\frac{m}{2\hbar\epsilon}\left(q_{j+1}^2 + \frac{1}{j} q_I^2\right).$$

Now use the Gaussian result

$$\int dx \exp\left(-ax^2 + bx + c\right) = \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2}{4a} + c\right), \qquad \text{if} \qquad \operatorname{Re} a > 0,$$

to give

$$\mathcal{I}_1^E = \sqrt{\frac{j}{j+1}} \exp\left[\frac{m}{2\hbar\epsilon}\left(\frac{j}{j+1}\left[q_{j+1} + \frac{1}{j} q_I\right]^2 - \left[q_{j+1}^2 + \frac{1}{j} q_I^2\right]\right)\right].$$

Let's now focus on the bit inside round brackets:

$$\begin{aligned}\frac{j}{j+1}\left(q_{j+1}+\frac{1}{j}q_I\right)^2-\left(q_{j+1}^2+\frac{1}{j}q_I^2\right) &= q_{j+1}^2\left(\frac{j}{j+1}-1\right)+\frac{q_I^2}{j}\left(\frac{1}{j+1}-1\right)-\frac{2q_{j+1}q_I}{(j+1)}\\ &= -\frac{1}{j+1}\left(q_{j+1}^2+q_I^2-2q_{j+1}q_I\right)\\ &= -\frac{1}{j+1}(q_{j+1}-q_I)^2,\end{aligned}$$

so if we undo our Wick rotation, i.e. $\epsilon \to i\epsilon$, we get

$$\mathcal{I}_1 = \sqrt{\frac{j}{j+1}}\exp\left(\frac{im}{2\hbar\epsilon}\frac{1}{j+1}(q_{j+1}-q_I)^2\right).$$

So now we consider the dq_2 integral: the square root factor obviously just factors out and the rest is simply

$$\exp\left[\frac{im}{2\hbar\epsilon}\left(\left(q_{j_2+1}-q_{j_2}\right)^2+\frac{1}{j_1+1}\left(q_{j_1+1}-q_I\right)^2\right)\right],$$

then we use $j_2 = j_1 + 1$ to give

$$\mathcal{I}_2 = \sqrt{\frac{j_1}{j_2}}\sqrt{\frac{-im}{2\pi\hbar\epsilon}}\int dq_2 \exp\left[\frac{im}{2\hbar\epsilon}\left(\left(q_{j_2+1}-q_{j_2}\right)^2+\frac{1}{j_2}\left(q_{j_2}-q_I\right)^2\right)\right],$$

which is (up to the first root factor) the equation we started the proof with. So we see the formula holds inductively (that's the reason we've been using j all the time rather then setting $j = 1$).

So we will just get a $\sqrt{j/j+1}$ factor for each integral, there's $n-2$ of these giving us the full result

$$K(q_F,T;q_I,0) = \lim_{n\to\infty}\sqrt{\frac{-im}{2\pi\hbar\epsilon}}\left[\prod_{j=1}^{n-2}\sqrt{\frac{j}{j+1}}\right]\exp\left(\frac{im}{2\hbar\epsilon}\frac{1}{n-1}(q_{j-1}-q_I)^2\right),$$

then we finally use $q_{j-1} = q_F$ which gives us exactly the result in the claim. □

Now we note that the product in the above formula will simply give

$$\prod_{j=1}^{n-2}\sqrt{\frac{j}{j+1}} = \frac{1}{\sqrt{n-1}},$$

and so, using $T = n\epsilon \approx (n-1)\epsilon$, where the second term is understood in the limit $n \to \infty$, we finally get

$$K(q_F,T;q_I,0) = \sqrt{\frac{-im}{2\pi\hbar T}}\exp\left(\frac{im}{2\hbar T}(q_F-q_I)^2\right) \propto \exp\left(\frac{i}{\hbar}S[q_{cl}]\right), \tag{3.3}$$

where the second part comes from the fact that the square-root term is independent of q_F/q_I and the definition of the classical action.

3.2.2 Simple Harmonic Oscillator

Another case we can solve exactly is the simple harmonic oscillator. The Lagrangian is

$$L = \frac{m}{2}\dot{q}^2 - \frac{m}{2}\omega^2 q^2.$$

This has classical action

$$S[q_{cl}] = \frac{m\omega}{2\sin(\omega T)}\Big[(q_F^2 + q_I^2)\cos(\omega T) - 2q_F q_I\Big]. \tag{3.4}$$

We can treat this as a variation to the classical path, and so as per the section above our path integral is

$$\begin{aligned}\int Dq e^{\frac{i}{\hbar}S[q]} &= e^{\frac{i}{\hbar}S[q_{cl}]}\int D\eta\, \exp\left(\frac{im}{2\hbar}(\dot{\eta}^2 - \omega^2\eta^2)\right)\\ &= e^{\frac{i}{\hbar}S[q_{cl}]}\int D\eta\, \exp\left[\frac{im}{2\hbar}\eta\Big(-\frac{d^2}{dt^2} - \omega^2\Big)\eta\right],\end{aligned}$$

where the second line follows from integration by parts.

Exercise

Prove that integration by parts leads to the expression above.

We then use our simplified notation Equation (3.1) and note that this is an eigenvector of the operator in our integral, i.e.

$$\Big(-\frac{d^2}{dt^2} - \omega^2\Big)\eta = \Big(\frac{\lambda^2\pi^2}{T^2} - \omega^2\Big)\eta,$$

to obtain (absorbing everything else into the proportionality constant, we will get it all back at the end)

$$\int Dq e^{\frac{i}{\hbar}S[q]} \propto \lim_{n\to\infty}\left[\prod_{j=1}^{n}\int da_\lambda\right]\exp\left[\frac{im}{2\hbar}\left(\frac{\lambda^2\pi^2}{T^2} - \omega^2\right)\sum_{i=1}^{n}a_\lambda^2\right].$$

Then we notice this is just the product of a bunch of Gaussian integrals, so we get

$$\int Dq\, e^{\frac{i}{\hbar}S[q]} \propto \prod_{\lambda=1}^{\infty}\left(\frac{\lambda^2\pi^2}{T^2} - \omega^2\right)^{-1/2} \propto \prod_{\lambda=1}^{\infty}\left(1 - \frac{\omega^2T^2}{\lambda^2\pi^2}\right)^{-1/2} = \left(\frac{\sin(\omega T)}{\omega T}\right)^{-1/2}.$$

Now the proportionality constant appears to have got very complicated as we've gone along, however it can easily be checked that we haven't removed anything that is a function of ω, and so we know that our path integral is of the form

$$\int D\eta\, e^{\frac{i}{\hbar}S[\eta]} = f(T)\left(\frac{\sin(\omega T)}{\omega T}\right)^{-1/2}.$$

So how do we find $f(T)$, well we use

$$\lim_{\omega\to 0}\left(\frac{\sin(\omega T)}{\omega T}\right) = 1,$$

to notice that in this limit our classic action, Equation (3.4), is just that of a free particle action, which gives us

$$\lim_{\omega\to 0}\int Dq\, e^{\frac{i}{\hbar}S[q]} = f(T)\exp\left(\frac{im}{2\hbar T}(q_F-q_I)^2\right),$$

which comparing to Equation (3.3) gives us

$$f(T) = \sqrt{\frac{-im}{2\pi\hbar T}},$$

and finally

$$\int Dq\, e^{\frac{i}{\hbar}S[q]} = \sqrt{\frac{-im\omega}{2\pi\hbar T\sin(\omega T)}}\exp\left(\frac{m\omega}{2\sin(\omega T)}\Big[(q_F^2+q_I^2)\cos(\omega T)-2q_Fq_I\Big]\right).$$

3.3 Schwinger-Dyson Equation

As we said before, it is not normally true that we can solve path integrals exactly, and we need some method to solve them perturbatively. We can gain an appreciation of this fact by finding the differential equations that the partition function satisfies. Recall that the partition function can be written

$$Z[J] = \mathcal{N}\int Dq\exp\left(\frac{i}{\hbar}S[q]+\int dtq(t)J(t)\right).$$

In this expression, $q(t)$ is just an integration variable, and so the result is completely unchanged if we change variables to

$$q(t)\to q(t)+\eta(t),$$

provided $\eta(t)$ is independent of q. That is, we also have (using $D(q+\eta)=Dq$ for fixed η)

$$\begin{aligned}Z[J] &= \mathcal{N}\int Dq\exp\left(\frac{i}{\hbar}S[q+\eta]+\int dtq(t)J(t)+\int dt\eta(t)J(t)\right)\\ &= \mathcal{N}\int Dq\exp\left(\frac{i}{\hbar}S[q]+\frac{i}{\hbar}\int dt\eta(t)\frac{\delta S[q]}{\delta q(t)}+\int dtq(t)J(t)+\int dt\,\eta(t)\frac{\delta}{\delta q(t)}\int dsJ(s)q(s)\right)+\mathcal{O}(\eta^2),\end{aligned}$$

where the second line simply comes from the expanding $S[q+\eta]$ and rewriting

$$\int dt\,\eta(t)J(t) = \int dt\,\eta(t)\frac{\delta}{\delta q(t)}\int dsJ(s)q(s).$$

Now the difference between the two different expressions for $Z[J]$ must vanish (as they are equal, all we've done is change integration variable). So if we take them away from each other and Taylor expand we get

$$\begin{aligned}0 &= \frac{i}{\hbar}\int dt\,\eta(t)\frac{\delta S[q]}{\delta q(t)}+\int dt\,\eta(t)\frac{\delta}{\delta q(t)}\int ds\,J(s)q(s)+\mathcal{O}(\eta^2)\\ &= \int dt\eta(t)\frac{\delta}{\delta q(t)}\left(\frac{i}{\hbar}S[q]+\int dsJ(s)q(s)\right)+\mathcal{O}(\eta^2)\\ &= \mathcal{N}\int Dq\int dt\eta(t)\frac{\delta}{\delta q(t)}\exp\left(\frac{i}{\hbar}S[q]+\int dsJ(s)q(s)\right)+\mathcal{O}(\eta^2),\end{aligned}$$

where the last line follows from the fact that the exponential is non-degenerate and that the path integral of 0 is 0. Now we use that we haven't specified the form of $\eta(t)$, and so we can conclude

$$0 = \mathcal{N} \int Dq \frac{\delta}{\delta q(t)} \exp\left(\frac{i}{\hbar} S[q] + \int ds J(s) q(s)\right).$$

Now we shall assume our Lagrangians are of the form

$$L(q, \dot{q}) = \frac{m}{2}\dot{q}^2 + V(q),$$

where $V(q)$ is some polynomial function. Then using the fact that the functional variation of the action gives us the Euler-Lagrange equations, the above condition become

$$0 = \mathcal{N} \int Dq \left(\frac{i}{\hbar}\Big[- V'\big(q(t)\big) - m\ddot{q}(t)\Big] + J(t)\right) \exp\left(\frac{i}{\hbar} S[q] + \int ds J(s) q(s)\right).$$

Finally we notice that

$$\frac{\delta}{\delta J(t)} \exp\left(\frac{i}{\hbar} S[q] + \int ds J(s) q(s)\right) = q(t),$$

and so we can replace the qs in the previous equation with functional derivatives w.r.t. $J(t)$. This allows us to pull these terms outside the path integral (i.e. past the Dq), giving us a differential equation for the partition function, known as the *Schwinger-Dyson equation*:

$$\left(- \frac{i}{\hbar}\left[V'\left(\frac{\delta}{\delta J(t)}\right) + m\frac{d^2}{dt^2}\frac{\delta}{\delta J(t)}\right] + J(t)\right) Z[J] = 0. \tag{3.5}$$

This now explains to us why the previous two problems were solvable: if we have a quadratic action, then V' is linear and so we get 2^{nd} order linear functional differential equation, which we can solve. However for the more general cases, when V' is not linear, things get a lot more complicated.

4 | SD For SHO & Perturbation Theory

4.1 Schwinger-Dyson For SHO

We ended last lecture the the derivation of the Schwinger-Dyson equation. We start this lecture by showing how it works for the example of a SHO system. Here the potential is

$$V(q) = \frac{m}{2}\omega^2 q^2, \qquad \Longrightarrow \qquad V'(q) = m\omega^2 q,$$

and so we get a nice linear equation for the Schwinger-Dyson formula:

$$\left[-\frac{im}{\hbar}\left(\omega^2 + \frac{d^2}{dt^2}\right)\frac{\delta}{\delta J(t)} + J(t)\right] Z[J] = 0.$$

Using the ansatz $Z[J] = e^{W[J]}$,[1] we get

$$\frac{im}{\hbar}\left(\frac{d^2}{dt^2} + \omega^2\right)\frac{\delta W[J]}{\delta J(t)} = J(t),$$

so if we take a second functional derivative w.r.t. $J(s)$, we get

$$\frac{im}{\hbar}\left(\frac{d^2}{dt^2} + \omega^2\right)\frac{\delta^2 W[J]}{\delta J(s)\delta J(t)} = \delta(s-t). \tag{4.1}$$

Now we use the definition of the partition function as the sum of N-point Green's functions along with the result

$$G(t) = \langle \Omega |\, q(t)\, | \Omega \rangle = 0$$

for the SHO we get

$$\begin{aligned} W[J] &= \ln Z \\ &= \ln\left(1 + \frac{1}{2}\int dtdsJ(t)J(s)G(t,s) + \widetilde{W}[J]\right) \\ &= \frac{1}{2}\int dtdsJ(t)J(s)G(t,s) + \widetilde{W}[J], \end{aligned}$$

[1] Here we have used the fact that $Z[0] = \langle 0|0\rangle = 1$ for a normalisable theory, otherwise we have to use $Z[J] = Ae^{W[J]}$.

where

$$\widetilde{W}[J] = \sum_{n=3}^{\infty} \frac{1}{n!} \int dt_1 ... dt_n J(t_1) ... J(t_n) G(t_1, ..., t_n) \tag{4.2}$$

and we have used

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + ...$$

So we conclude that the 2-point Green's function is indeed a Green's function with the operator

$$\frac{im}{\hbar}\left(\frac{d^2}{dt^2} + \omega^2\right) G(t,s) = \delta(s-t). \tag{4.3}$$

We use our intuition from solving differential equations to suggests the ansatz

$$G(t,s) = Ce^{-i\omega|s-t|},$$

where we take the modulus as we know that $G(t,s) = G(s,t)$. We can actually go one step further and use the other known fact that

$$G(0,0) = \langle \Omega | \, q^2 \, | \Omega \rangle = \frac{\hbar}{2m\omega} \tag{4.4}$$

for the SHO to guess

$$G(t,s) = \frac{\hbar}{2m\omega} e^{\pm i\omega|s-t|}.$$

Ok let's check this does indeed satisfy Equation (4.3). We start by rewriting our ansatz as

$$G(t,s) = \begin{cases} A_+ \cos(|s-t|\omega) + B_+ \sin(|s-t|\omega) & s > t \\ A_- \cos(|s-t|\omega) + B_- \sin(|s-t|\omega) & t > s. \end{cases}$$

It follows immediately from the symmetry $G(t,s) = G(s,t)$ that we require

$$A_+ = A_- \quad \text{and} \quad B_+ = B_-,$$

so we can just write

$$G(t,s) = A\cos(|s-t|\omega) + B\sin(|s-t|\omega)$$

We get the value of A straight from Equation (4.4),

$$A = G(0,0) = \frac{\hbar}{2m\omega}.$$

Now we want the action of the derivative operator in Equation (4.3) to give a delta function, which is discontinuous. Therefore, using

$$\cos|x| = \cos x,$$

which is smooth,[2] we know that it is the sin term that gives us it. Now away from $s = t$ we have a smooth result, so we only need look at the region $|s-t| \approx 0$, and therefore we can Taylor expand:

$$\frac{im}{\hbar}\left(\frac{d^2}{dt^2} + \omega^2\right) G(t,s) = \frac{im}{\hbar}\left(\frac{d^2}{dt^2} + \omega^2\right) B|s-t|\omega + \mathcal{O}(\omega^3)$$

[2] That is infinitely differentiable, with continuous result.

Now we claim that we don't need to consider the ω^2 part of the operator as this will cancel with the $\mathcal{O}(\omega^3)$ term in the expansion.[3] Then using the fact

$$\frac{d}{dx}|x| = \frac{x}{|x|} = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases} \qquad \Longrightarrow \qquad \frac{d^2}{dx^2}|x| = 2\delta(x),$$

we get

$$\frac{im}{\hbar}\left(\frac{d^2}{dt^2} + \omega^2\right)G(t,s) = \frac{im}{\hbar}B\omega 2\delta(s-t),$$

so comparing to Equation (4.3) we conclude

$$B = -\frac{i\hbar}{2m\omega}$$

So altogether we have

$$G(t,s) = \frac{\hbar}{2m\omega}\big(\cos(|t-s|\omega) - i\sin(|t-s|\omega)\big) = \frac{\hbar}{2m\omega}e^{-i|s-t|\omega}. \tag{4.5}$$

Now what about $\widetilde{W}[J]$? Well we need it to satisfy

$$\frac{im}{\hbar}\left(\frac{d^2}{dt^2} + \omega^2\right)\frac{\delta^2\widetilde{W}[J]}{\delta J(s)\delta J(t)} = 0,$$

otherwise we would get another term on the right-hand side of Equation (4.1). Using the definition Equation (4.2) and the above calculation it's clear we'll get something of the form

$$G(t_1,...,t_n) = \big(A_1\cos(\omega t_1) + B_1\sin(\omega t_1)\big)...\big(A_n\cos(\omega t_n) + B_n\sin(\omega t_n)\big),$$

but we also require that $G(t_1,...,t_n)$ is totally symmetric, so we also require

$$A_1 = ... = A_n = B_1 = ... = B_n.$$

We then also need the result to be time-translation invariant, but this is not possible here and so we have to conclude that the coefficients all vanish. This just leaves us with

$$W[J] = \frac{1}{2}\int dtdsJ(t_1)J(t_2)G(t_1,t_2),$$

and so we conclude that the partition function is[4]

$$Z_0[J] = \exp\left(\frac{1}{2}\int dt_1dt_2J(t_1)G_0(t_1,t_2)J(t_2)\right), \tag{4.6}$$

with $G_0(t_1,t_2)$ given by Equation (4.5).

[3] That is we would get a term $\frac{d^2}{dt^2}\big(-\frac{\omega^3}{3!}|s-t|^3\big)$, which cancels the $\omega^2|s-t|\omega$ term.

[4] The subscripts 0 will make sense in a minute.

4.2 Perturbation Theory

In order to make the perturbation calculations easier to follow, we introduce some new notation, as listed below.

- $J_1 = J_{t_1} := J(t)$,
- $(G_0)_{t_1,t_2} := G_0(t_1, t_2)$,
- $J \cdot G_0 \cdot J := \int dt_1 dt_2 J_1 (G_0)_{t_1,t_2} J_2$,
- $(J \cdot G_0)_2 := \int dt_1 J_1 (G_0)_{t_1,t_2}$, and similarly for $(G_0 \cdot J)_1$.

Now let's suppose we want to study some system that can be expressed as a perturbation around some known solution, that is the Lagrangian is of the form

$$L(q, \dot{q}) = L_0(q, \dot{q}) - V(q),$$

where $L_0(q, \dot{q})$ is our known system and $V(q)$ is a 'small' perturbation (i.e. the coupling constant that appears in it is small). The partition function for the known system is given by

$$Z_0[J] = \mathcal{N}_0 \int Dq \, \exp\left(\frac{i}{\hbar} S_0[q] + \int dt J(t) q(t)\right)$$

where $\mathcal{N}_0$ is some normalisation constant. The full theory then has partition function

$$Z[J] = \widetilde{\mathcal{N}} \int Dq \exp\left(\frac{i}{\hbar} S_0[q] + \int dt J(t) q(t) - \frac{i}{\hbar} \int dt V\big(q(t)\big)\right).$$

We then employ our clever trick used in the derivation of the Schwinger-Dyson equation and trade the argument $q(t)$ of V for a functional variation w.r.t. $J(t)$ and pull it outside the path integral, giving us

$$\begin{aligned} Z[J] &= \exp\left[-\frac{i}{\hbar} \int dt V\left(\frac{\delta}{\delta J(t)}\right)\right] \widetilde{\mathcal{N}} \int Dq \exp\left(\frac{i}{\hbar} S_0[q] + \int dt J(t) q(t)\right) \\ &= \mathcal{N} \exp\left[-\frac{i}{\hbar} \int dt V\left(\frac{\delta}{\delta J(t)}\right)\right] Z_0[J], \end{aligned}$$

where $\mathcal{N} = \widetilde{\mathcal{N}}/\mathcal{N}_0$. Now note that $Z_0[J]$ is independent of the coupling constants (i.e. the parameters in $V(q)$) and so we get a perturbation series by expanding the exponential on the last line above to the required power of coupling.[5]

Ok this is all rather abstract, so let's look at an example.

Example 4.2.1. Let

$$L = \underbrace{\frac{m}{2}\dot{q}^2 - \frac{m}{2}\omega^2 q^2}_{L_0 = L_{SHO}} \underbrace{- \frac{\lambda}{3!} q^3}_{V(q)},$$

[5] If this idea of 'power of coupling' doesn't mean anything to you, hopefully it will make sense in a moment. If not, reading an introduction to QFT from the second-quantisation approach and Feynman diagrams should clear up any confusion.

where we have identified the simple harmonic oscillator as our known system. For clarity with the footnote above, λ here is our coupling constant and we take it to be small. Roughly speaking, it corresponds to the coupling strength for 3 particles interact (as it comes with q^3). In the Taylor expansion the λ^n term will come with q^{3n}, and so corresponds to the interaction of $3n$ particles, and this is what we mean by expanding to the relevant power of the coupling constant — just truncate your expansion to the order of particle interactions you want to consider.

We have already found the partition function for the SHO, Equation (4.6), and so (using the notation introduced at the start of this section) the partition function for the full theory is

$$Z[J] = \mathcal{N} \exp\left[-\frac{i}{\hbar}\frac{\lambda}{3!}\int dt\left(\frac{\delta^3}{\delta J^3(t)}\right)\right] \exp\left(\frac{1}{2}J\cdot G_0\cdot J\right).$$

Let's just consider coupling to first order, we therefore get

$$Z[J] = \mathcal{N}\left[1 - \frac{i}{\hbar}\frac{\lambda}{3!}\int dt\left(\frac{\delta^3}{\delta J^3(t)}\right) + \mathcal{O}(\lambda^2)\right] \exp\left(\frac{1}{2}J\cdot G_0\cdot J\right).$$

We need to find the functional derivatives, and we find this by considering them to be analogous to 'regular derivatives' and use the chain rule:[6]

$$\begin{aligned}
\frac{\delta^3}{\delta J^3(t)} \exp\left(\frac{1}{2}J\cdot G_0\cdot J\right) &= \frac{\delta^2}{\delta J^2(t)}\left[(J\cdot G_0)_t \exp\left(\frac{1}{2}J\cdot G_0\cdot J\right)\right] \\
&= \frac{\delta}{\delta J(t)}\left[\left((G_0)_{t,t} + (J\cdot G_0)_t^2\right)\exp\left(\frac{1}{2}J\cdot G_0\cdot J\right)\right] \\
&= \left[(G_0)_{t,t}(J\cdot G_0)_t + 2(G_0)_{t,t}(J\cdot G_0)_t + (J\cdot G_0)_t^3\right]\exp\left(\frac{1}{2}J\cdot G_0\cdot J\right) \\
&= \left[3(G_0)_{t,t}(J\cdot G_0)_t + (J\cdot G_0)_t^3\right]\exp\left(\frac{1}{2}J\cdot G_0\cdot J\right).
\end{aligned}$$

So plugging this into our expression for the partition function, we get

$$Z[J] = \mathcal{N}\left[1 - \frac{i}{\hbar}\lambda\int dt\left(\frac{1}{2}(G_0)_{t,t}(J\cdot G_0)_t + \frac{1}{3!}(J\cdot G_0)_t^3\right)\right]\exp\left(\frac{1}{2}J\cdot G_0\cdot J\right). \tag{4.7}$$

Finally we can obtain the normalisation coefficient by looking at the 0-point function (i.e. $Z[0]$) and requiring it be one, This gives

$$1 = \mathcal{N}(1+0)e^0 \qquad \Longrightarrow \qquad \mathcal{N} = 1 + \mathcal{O}(\lambda^2).$$

4.2.1 Feynman Diagrams

To those familiar with the Hamiltonian approach to QFT, the title of this subsection is probably a huge relief. We can indeed get some Feynman rules from the path integral approach, however one should note that these rules are specific to this case, and so some symbols will

[6] Note by the symmetry of G_0 we have $(J\cdot G_0)_{t_2} = (G_0\cdot J)_{t_1}$, so we can remove the factors of 1/2 under derivative.

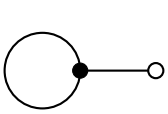
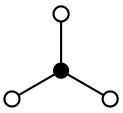
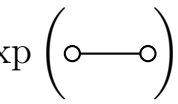
appear that you do not see in the second-quantisation Feynman diagrams. We use the above example as a proxy to state the Feynman rules and leave it to the reader to imagine how they adapt to other cases (it is exactly as you would think).

Part Of Diagram	Maths Expression	Comment
	$G_0(t_1, t_2)$	t_1/t_2 are the end points, usually we do not explicitly label them.[a]
	$\int ds J(s)$	The integrand includes the other factors from the diagram.
	$-\frac{i\lambda}{\hbar}\int dt$	This is the interaction vertex specifically for our example.[b] This clearly changes depending on the theory.

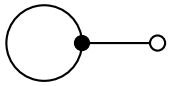
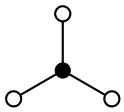
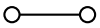
[a] They are always integrated over so just match to the integration variables.
[b] Note we don't put the 3! here, this comes from the symmetry factors, see blow.

As with the second-quantisation approach, we just draw all the possible Feynman diagrams we can at the relevant order of coupling and then obtain the maths expression using the rules above (and also accounting for the symmetry factors). Let's do the example for the previous example.

Example 4.2.2. We represent the above calculation in Feynman diagrams as

$$\frac{1}{\mathcal{N}}Z[J] = \left[1 + \text{  } + \text{  } + \mathcal{O}(\lambda^2)\right] \exp\left(\text{  }\right)$$

We can then work out the symmetry factors by looking at the diagrams:

Diagram	Symmetry Factor	Why
	$\frac{1}{2}$	We can reflect the loop about the horizontal axis.
	$\frac{1}{3!}$	Permutations of the three lines.
	$\frac{1}{2}$	Can reflect in a vertical line down the middle.

Exercise

Convince yourself that the diagram given above does indeed give us Equation (4.7).

4.2.2 Green's Functions From Feynman Diagrams

Recall that we can obtain the N-point Green's function by taking functional derivatives w.r.t. $J(t)$ of the partition function, Equation (2.6):

$$G(t_1,..,t_N) = \frac{\delta^N Z[J]}{\delta J(t_1)...\delta J(t_N)}\bigg|_{J=0}.$$

We now want to see how to do this in terms of our diagrams. Well the hollow circles represent integrated factors of $J(s)$ in the diagram, so our functional varitation will remove this circle and replace it with the derivative Js argument, i.e. we use, for example,

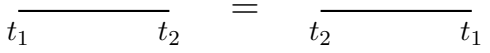
$$\frac{\delta^2}{\delta J_1 J_2}\left(\frac{1}{2}(J\cdot G_0\cdot J)_{s,r}\right) = (G_0)_{t_1,t_2}$$

to replace

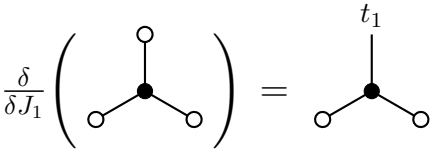
$$\frac{\delta^2}{\delta J_1 \delta J_2}\Big(\quad\Big) = \quad t_1 \qquad t_2$$

Note that it is only the terms that have exactly N hollow circles in it that survive this process: if there's less then there's a derivative that has nothing to act on; if there's more then setting $J=0$ corresponds to saying diagrams with a hollow circle left vanish.

Remark 4.2.3. Note that our differentiated diagrams need to respect the symmetry property of the maths. For example, we know that $(G_0)_{t_1,t_2} = (G_0)_{t_2,t_1}$ and so we must have



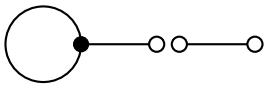
Note we don't get factors from this symmetry though, because the diagrams come with that same factor divided. For example



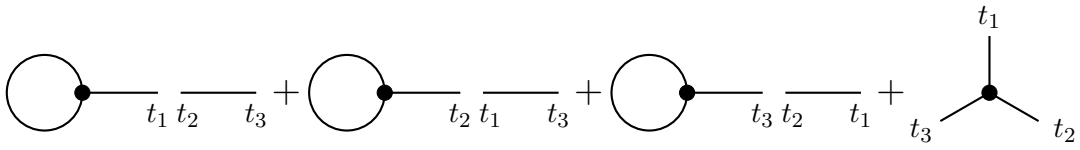
without a factor of 3! at the front. That is, there are 3! different ways to get the right-hand side, but the symmetry factor of this diagram is exactly 1/3!.

Example 4.2.4. We can find the 2-point function for our q^3 theory. To first order in λ, all the diagrams outside the exponential vanish (as none have exactly 2 hollow circles), and so we are just left with the terms that are pulled down from the action on the exponential. It is clear[7] that what we are left with at the end is simply

We now note that from the diagrams we can obtain terms that do not appear in the expansion of Equation (4.7). For example, the following diagram is a valid diagram at first order in λ



where we note this is *not* the sum of two diagrams, but is collectively one diagram. We see that this is essentially two of the previous diagrams 'put next to each other' in a disconnected way. This diagram will give a non-zero contribution to the 3-point Green's function, and it is easy to convince yourself (exercise below) that the full 3-point function is given by



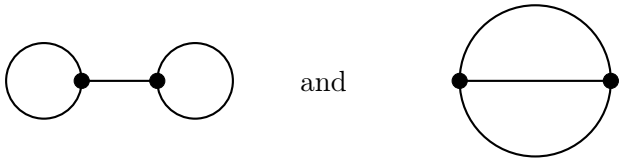
Exercise

Convince yourself that the above is indeed the 3-point function for the q^3 theory.

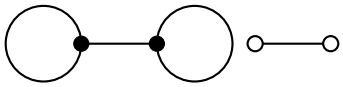
[7] And hence I'm saving myself the pain of Tikzing it all.

4.2.3 Vacuum Bubbles

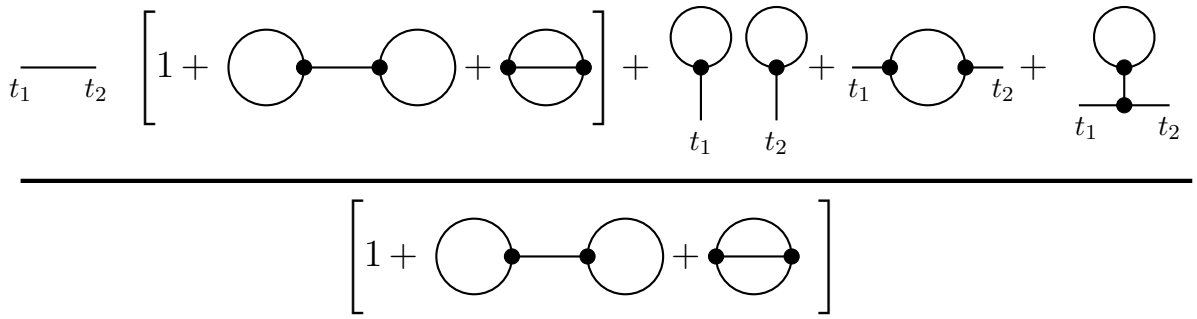
Now note that if we go to order λ^2 then we will get diagrams that look like



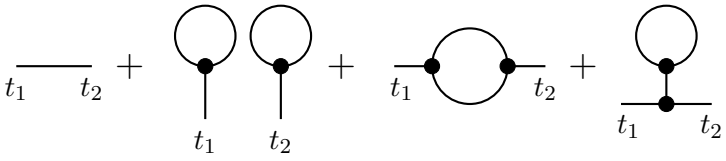
These diagrams have no external legs (i.e. there are no hollow circles), and we refer to diagrams of this kind as *vacuum bubbles*. Our partition function does not contain vacuum bubbles and so we need to 'divide the diagrams out'. What do we mean by this? Well we note that for every one of these vacuum bubble diagrams we also get the same diagram but now with a $(J \cdot G_0 \cdot J)$ term too, e.g.



We therefore 'factor out' the vacuum bubble parts and divide by the purely vacuum bubble diagrams to cancel them. For example, the 2-point function for the q^3 theory at order λ^3 is given by the division of diagrams



which after the division just leaves



You might protest and say "but the vacuum bubble diagrams didn't appear with the diagrams not in the square brackets above, so why do they cancel there?" The answer is the the denominator corresponds to

$$\frac{1}{1+\mathcal{O}(\lambda^2)} = 1 - \mathcal{O}(\lambda^2)$$

where the second line follows from λ being small. The diagrams in the numerator already contain λ powers and so to order λ^2 they only pick up the 1.

Remark 4.2.5. Note you can use the same argument for the terms that do cancel: here you get the vacuum bubble terms coming with a plus sign (bit in square bracket) and also with a minus sign (from binomial expansion above), and so they cancel.

This gives us the final general conclusion

> N-point function is given by sum of all diagrams with exactly N external legs but no vacuum bubble contributions.

Exercise

Convince yourself that there are no other contributions to the 2-point function at order λ^2.

4.3 Semi-Classical Expansion

We conclude the course with a brief discussion of the semi-classical approximation to the path integral. We obtain this by Taylor expanding the action about a classical path q_{cl}:

$$S[q_{cl}+\eta] = S[q_{cl}] + \int dt_1 \eta(t_1) \frac{\delta S}{\delta q(t_1)}[q_{cl}] + \frac{1}{2!}\int dt_1 dt_2 \eta(t_1)\eta(t_2)\frac{\delta^2 S}{\delta q(t_1)\delta q(t_2)}[q_{cl}] + ...,$$

now we note that the term with a single integral vanishes as the variation of the action gives the Euler-Lagrange equations and the classical path minimises these. Now let's consider rescalling

$$\eta \to \sqrt{\hbar}\eta,$$

and noting, as before,

$$D(q_{cl}+\eta) = D\eta,$$

then our Feynman kernel becomes

$$\begin{aligned} K(q_F,T;q_I,0) &= \int Dq \, \exp\left(\frac{i}{\hbar}S[q_{cl}+\eta]\right) \\ &\propto e^{\frac{i}{\hbar}S[q_{cl}]}\int D\eta \, \exp\left[\frac{i}{2!}\int dt_1 dt_2 \eta(t_1)\eta(t_2)\frac{\delta^2 S[q_{cl}]}{\delta q(t_1)\delta q(t_2)} + ... \right. \\ &\qquad \left. + \frac{i}{n!}\hbar^{\frac{n}{2}-1}\int dt_1 ... dt_n \eta(t_1)...\eta(t_n)\frac{\delta^n S[q_{cl}]}{\delta q(t_1)...\delta q(t_n)} + ...\right] \\ &= e^{\frac{i}{\hbar}S[q_{cl}]}\int D\eta \, \exp\left[\frac{i}{2!}\int dt_1 dt_2 \eta(t_1)\eta(t_2)\frac{\delta^2 S[q_{cl}]}{\delta q(t_1)\delta q(t_2)}\right]\Big(1+\mathcal{O}\big(\sqrt{\hbar}\big)\Big). \end{aligned}$$

Now obviously the prefactor is the classical contribution, and we see that the leading order quantum contribution is a Gaussian path integral in η, i.e. its of the form

$$\int D\eta \, \exp\big(i\eta\cdot\Omega\cdot\eta\big) \propto \big(\det(-i\Omega)\big)^{-1/2},$$

where[8]

$$\begin{aligned}\Omega_{t_1,t_2} &= \frac{1}{2}\frac{\delta^2 S[q_{cl}]}{\delta q(t_1)\delta q(t_2)} \\ &= -\frac{1}{2}\left(m\frac{d^2}{dt_1^2} + V''(q_{cl})\right)\delta(t_1 - t_2)\end{aligned}$$

where we have assumed the Lagrangian has the usual kinetic term (i.e. $\frac{m}{2}\dot{q}^2$). $\det(-i\Omega)$ is the product of the eigenvalues of the operator

$$\frac{i}{2}\left(m\frac{d^2}{dt_1^2} + V''(q_{cl})\right)$$

acting on $\eta(t)$ subject to the constraints $\eta(t_I) = \eta(t_F) = 0$. The semiclassical approximation is when we ignore the contributions from the $\mathcal{O}(\sqrt{\hbar})$ terms.

Remark 4.3.1. If the action is quadratic in q (e.g. the SHO) then the semiclassical solution is exact.

Remark 4.3.2. If the classical path is not unique we must take a sum over all the different classical solutions.

Example 4.3.3. Let's look at the SHO as an example. As just remarked, this will give us an exact result. Here we have

$$V''(q_{cl}) = m\omega^2, \qquad \Longrightarrow \qquad \Omega_{t_1,t_2} = -\frac{m}{2}\left(\frac{d^2}{dt_1^2} + \omega^2\right)\delta(t_1 - t_2).$$

The eigenvectors of the operator here are

$$\eta(t) = A\cos(\widetilde{\omega}t) + B\sin(\widetilde{\omega}t),$$

with eigenvalues

$$\lambda = \omega^2 - \widetilde{\omega}^2.$$

If we now impose the condition $\eta(t_I) = 0 = \eta(t_F)$ with $t_I = 0$, $t_F = T$, we conclude

$$\widetilde{\omega} = \frac{n\pi}{T}, \qquad n \in \mathbb{N},$$

and so

$$\lambda = \omega^2 - \left(\frac{n\pi}{T}\right)^2 \propto 1 - \left(\frac{\omega T}{n\pi}\right)^2.$$

We can therefore conclude

$$\big(\det(-i\Omega)\big)^{-1/2} \propto \prod_{n=1}^{\infty}\left[1 - \left(\frac{\omega T}{n\pi}\right)^2\right]^{-1/2} = \sqrt{\frac{\omega T}{\sin(\omega T)}}.$$

[8] Extra exercise: prove this formula holds.

Useful Texts & Further Readings

QFT

- Zee, Anthony. *Quantum Field Theory In A Nutshell*. Vol. 7. Princeton University Press, 2010.